ZEN18

UN

WHAT IS THE LAPLACE TRANSFORM?

D. V. WIDDER, Harvard University

1. Introduction. A *Laplace integral* is an integral of the form

$$f(s) = \int_0^\infty e^{-st}\phi(t)dt, \tag{1}$$

where $\phi(t)$ is any function which, for some value of s, gives the integral meaning. The integral then exists for a whole interval of values of s, so that a function $f(s)$ is defined. Since equation (1) may be thought of as *transforming* $\phi(t)$ into $f(s)$, it is frequently called the *Laplace transform*.

If, for example, $\phi(t)=1$, then

$$f(s) = \lim_{R\to\infty} \int_0^R e^{-st}dt = \lim_{R\to\infty} (1 - e^{-sR})/s \qquad s \neq 0. \tag{2}$$

When $s>0$ this limit is $1/s$; when $s<0$ there is no limit. More generally, if s is complex, $s=\sigma+i\tau$, the limit (2) exists or fails to exist according as σ is positive or negative. The general integral (1) behaves similarly: it *converges* in a right half-plane, $\sigma>\sigma_c$, and *diverges* for $\sigma<\sigma_c$. The number σ_c, which may be $+\infty$ or $-\infty$, is called the *abscissa of convergence*.

The name for the integral (1) was chosen because Laplace* used it extensively in his theory of probability. The modern revival of interest in the transform probably was caused by Riemann's discovery that the distribution of prime numbers depends upon the position of the zeros of the Zeta-function,

$$\zeta(s) = \sum_{n=1}^\infty \frac{1}{n^s}. \tag{3}$$

His famous conjecture, still unverified, that any zero of this function with positive real part must have the real part $1/2$ has had a tremendous influence on the development of mathematics. Now $\zeta(s)/s$ is a Laplace integral. Accordingly it was very natural that in looking for specific properties of the special function (3) the general properties of the integral (1) should have been discovered.

In this brief note no attempt will be made to give references for the various results described. They may be found in the treatises listed in the bibliography.

2. Relation to power series. A natural way of generalizing a power series

$$F(z) = \sum_{n=0}^\infty a_n z^n \tag{4}$$

is to replace the integral exponent n by an arbitrary real number λ_n. But for complex z the function z^{λ_n} usually has many values. One convenient way of specifying which value is intended is to set $z=e^{-s}$,

* See, for example, pages 111 to 180 in volume VII of the collected works of Laplace.

$$F(e^{-s}) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n s}. \tag{5}$$

If λ_n tends to $+\infty$, this is a *Dirichlet series*. The Zeta-function is seen to be a special case of the series (5) by taking $\lambda_n = \log n$.

It is now natural to generalize a step further by changing λ_n to a continuous variable t. The summation becomes an integral, the sequence a_n becomes a function $\phi(t)$, and the Dirichlet series becomes the Laplace integral. Accordingly it is not surprising that many of the properties of the integral can be correctly conjectured from the corresponding ones for power series. For example, the region of convergence of a power series is $|z| < \rho$. Hence we might expect that the region of convergence of a Laplace integral is $|e^{-s}| = e^{-\sigma} < \rho$ or $\sigma > -\log \rho$, a right half-plane. We have stated this fact and verified it in a particular example.

Let us list several properties that do carry over from power series:

	Power series	Laplace integrals
1. Convergence	$\lvert z\rvert < \rho$	$\sigma > \sigma_c$
2. Differentiation	$F'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$	$f'(s) = -\int_0^{\infty} e^{-st} t\phi(t)\,dt$
3. Analyticity	$\lvert z\rvert < \rho$	$\sigma > \sigma_c$
4. Uniqueness	$F \equiv 0$ implies $a_n \equiv 0$	$f \equiv 0$ implies $\phi(t) \equiv 0$
5. Inversion	$a_n = \frac{1}{2\pi i}\int_{\lvert z\rvert = k} F(z) z^{-n-1}\,dz$	$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} f(s) e^{st}\,ds$
6. Products	$FG = \sum_{n=0}^{\infty} c_n z^n$	$fg = \int_0^{\infty} e^{-st}\omega(t)\,dt$
	$c_n = \sum_{k=0}^{n} a_k b_{n-k}$	$\omega(t) = \int_0^t \phi(x)\psi(t-x)\,dx$

In property 4. it must be understood that $\phi(t)$ is *essentially* zero. It may be different from zero at some points, but at any rate its integral over any interval must vanish. In 5. $k < \rho$ and $c > \sigma_c$. Here the analogy is not quite complete since the substitution $z = e^{-s}$ carries the circle $|z| = k$ into only a piece of the vertical line $\sigma = c = -\log k$. In 6. the power series expansion of $G(z)$ has coefficients b_n and $g(s)$ is the Laplace transform of $\psi(t)$.

It is also important to observe certain fundamental differences between the two theories. Whereas a power series converges absolutely inside its region of convergence the same is not true for Laplace integrals. Indeed for the latter there are abscissas of conditional, absolute and uniform convergence, generally all different. Series (4) converges out to the singularity of $F(z)$ nearest the origin; $f(s)$ need have no singularity on the line $\sigma = \sigma_c$. Again, every analytic function has a power series expansion; the function $f(s) = s$ is entire but is not the Laplace transform of any function. Finally, there is invariably a difference in the meth-

ods of proof in the two theories. Often the gap is bridged by an integration by parts, for this process replaces a conditionally convergent Laplace integral by one which converges absolutely.

3. The bilateral transform. If the lower limit of integration in the integral (1) is replaced by $-\infty$, the new equation defines the *bilateral Laplace transform*. It is analogous to a Laurent series, and its region of convergence is a vertical strip. An important example is the Gamma-function,

$$\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}dx = \int_{-\infty}^\infty e^{-st}e^{-e^{-t}}dt.$$

The region of convergence is $\sigma>0$; the vertical strip has been enlarged into a half-plane. The inversion formula 5. applies as well to the bilateral transform.

4. Relation to the Fourier transform. Set $s=iy$ in a bilateral transform,

$$g(y) = f(iy) = \int_{-\infty}^\infty e^{-iyt}\phi(t)dt.$$

This is the Fourier transform of $\phi(t)$ into $g(y)$. It is thus a bilateral Laplace transform considered along a single vertical line. However, it becomes most useful when the strip of convergence reduces to a line. The Laplace theory is then fairly vacuous. By setting $c=0$ and $s=iy$ the inversion formula 5. becomes

$$\phi(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} f(s)e^{st}ds = \frac{1}{2\pi}\int_{-\infty}^\infty g(y)e^{iyt}dy.$$

This is the usual inversion of the Fourier transform; but of course it is not usually a corollary of formula 5. since for its validity the line of integration must lie *inside* a strip of convergence.

5. A table of transforms. It is useful to have a table of Laplace transforms to be used like any table of integrals. Here is a highly abridged one:

	$f(s)$	$\phi(t)$	σ_a
A.	$\frac{1}{s}$	1	0
B.	$\frac{1}{s^{n+1}}$	$\frac{t^n}{n!}$	0
C.	$\frac{1}{s-a}$	e^{at}	a
D.	$\frac{1}{s^2+1}$	$\sin t$	0
E.	$\frac{s}{s^2+1}$	$\cos t$	0

The pair B. may be derived from A. by differentiation; C. comes from the same source when s is replaced by $s-a$. Here α is the real part of a. Both D. and E. come from C. when $\sin t$ and $\cos t$ are expressed in terms of e^{it} and e^{-it}.

6. Differential equations. As an illustration of the many applications of the general theory let us solve a differential equation. It will be clear from the example that the method is very general. The transform (1) applied to an ordinary linear differential equation with constant coefficients reduces it to an algebraic one. The solution of the latter is then retransformed by inversion, or by use of a table. More generally, if the original equation is partial in any number of independent variables, one application of the Laplace transform reduces the number of variables by one.

Let it be required to find a function $y(t)$ such that $y(0)=1$, $y'(0)=2$ and

$$y'' + y = 2e^t. \tag{6}$$

Denote the Laplace transform of $y(t)$ by $Y(s)$. Integration by parts gives

$$\int_0^\infty e^{-st}y''(t)dt = -y'(0) - y(0)s + s^2\int_0^\infty e^{-st}y(t)dt$$

on the assumption that the integrated part vanishes at $+\infty$, at least for large values of s. Using the pair C. of §5 we see that the transformed equation is

$$-2 - s + s^2Y(s) + Y(s) = \frac{2}{s-1}$$

$$Y(s) = \frac{s^2+s}{(s-1)(s^2+1)} = \frac{1}{s-1} + \frac{1}{s^2+1}.$$

A second reference to the table and an appeal to the uniqueness property 4. shows that $y(t)=e^t+\sin t$. This function has the required properties, so that it is unnecessary to check the assumption made about its behavior at $+\infty$.

The procedure is elementary and could be used in an introductory study of differential equations. The proof that the method always leads to the solution has hitherto depended on contour integration in the complex plane. In a forthcoming text on advanced calculus by the author a simpler proof depending only on real variable theory will appear.

7. Real inversion. The most familiar determination of the coefficients of the power series (4) is $a_n = F^{(n)}(0)/n!$. An analogous inversion of the Laplace transform (1) has recently been discovered. It is

$$\phi(t) = \lim_{k\to\infty} \frac{(-1)^k}{k!} f^{(k)}\left(\frac{k}{t}\right)\left(\frac{k}{t}\right)^{k+1}. \tag{7}$$

Observe that $\phi(t)$ is determined by the values of the successive derivatives of $f(s)$ near $s=+\infty$. Since $z=0$ corresponds to $s=+\infty$ when $z=e^{-s}$ formula (7)

is indeed similar to the Taylor "inversion" of a power series. For the pair $f(s)=s^{-2}$, $\phi(t)=t$ equation (7) becomes

$$t = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)t.$$

For the pair C. it is

$$e^{at} = \lim_{k\to\infty}\left(1-\frac{at}{k}\right)^{-k-1}.$$

8. Relation to the moment problem. The *moment problem* of Hausdorff is the determination of a non-decreasing function $\beta(x)$ such that

$$\mu_n = \int_0^1 x^n d\beta(x) \qquad n = 0, 1, 2, \cdots, \tag{8}$$

where the integral is a Stieltjes integral. Hausdorff showed that the problem has a solution if and only if, for all n,

$$\mu_n \geqq 0,\ \Delta\mu_n = \mu_{n+1} - \mu_n \leqq 0,\ \Delta^2\mu_n = \mu_{n+2} - 2\mu_{n+1} + \mu_n \geqq 0, \cdots.$$

Such a sequence is given by $\mu_n = 1/(n+1)$ or by $(1/2)^n$, and these arise from equation (8) by taking first $\beta(x)=x$ and then $\beta(x)=0$ or 1 according as $x<1/2$ or $x>1/2$. Both functions are non-decreasing.

If in equation (8) we replace n by a continuous variable s and set $x=e^{-t}$ we obtain

$$\mu(s) = \int_0^\infty e^{-st} d[-\beta(e^{-t})] \qquad 0 \leqq s < \infty.$$

But this is a *Laplace-Stieltjes transform*

$$f(s) = \int_0^\infty e^{-st} d\alpha(t). \tag{9}$$

By analogy with the Hausdorff result we might expect that a non-decreasing function $\alpha(t)$ would exist satisfying equation (9) if and only if

$$f(s) \geqq 0,\ f'(s) \leqq 0,\ f''(s) \geqq 0, \cdots \qquad 0 < s < \infty.$$

Such a function is called *completely monotonic*. Examples are $f(s)=1/(s+1)$ and $f(s)=e^{-s}$, and these arise from equation (9) by use of the non-decreasing functions $\phi(t)=e^{-t}$ and $\phi(t)=0$ or 1 according as $t<1$ or $t>1$. The conjectured result is true and is known as Bernstein's theorem. The result is particularly remarkable in view of the fact that the signs of the derivatives of a function on the real axis should determine not only its analyticity in a half-plane but its representation in the form (9).

At first sight it may not be easy to see why the Stieltjes integral has been introduced at this stage. It is done largely to produce elegant results like those

of Hausdorff and Bernstein given above. The function e^{-s} though completely monotonic, could never have a representation (1). The same is true of any convergent Dirichlet series with positive coefficients. It is only when we introduce the Stieltjes integral, thus combining the class of such Dirichlet series with the functions defined by equation (1) with $\phi(t) \geqq 0$, that we can obtain a neat characterization of completely monotonic functions.

9. The Stieltjes transform. If the transform (1) is applied to itself there results, after a change in the order of integration,

$$f(s) = \int_0^\infty e^{-sx}\,dx \int_0^\infty e^{-xt}\,\phi(t)dt = \int_0^\infty \frac{\phi(t)}{s+t}\,dt. \tag{10}$$

This is called the *Stieltjes transform* because it was put to effective use by Stieltjes in his theory of continued fractions. A sample pair is $f(s) = 1/(\pi\sqrt{s})$, $\phi(t) = 1/\sqrt{t}$. This may be verified by formula 482 of B. O. Peirce's table of integrals. From the origin of the transform as an *iterated* Laplace transform many of its properties can be conjectured. An inversion formula related to property 5. of §3 is

$$\phi(t) = \lim_{\epsilon \to 0} \frac{f(-t - i\epsilon) - f(-t + i\epsilon)}{2\pi i}. \tag{11}$$

Another, matching the one given in §7, is

$$\phi(t) = \lim_{k \to \infty} \frac{(-t)^{k-1}}{k!(k-2)!} \left[t^k f(t)\right]^{(2k-1)}. \tag{12}$$

It is interesting to test this equation with the pair of transforms given above.

A result analogous to Bernstein's is that a non-negative number P and a non-decreasing function $\alpha(t)$ such that

$$f(s) = P + \int_0^\infty \frac{d\alpha(t)}{s+t}$$

exist if and only if

$$f(s) \geqq 0, \quad (-1)^{k-1}\left[s^k f(s)\right]^{(2k-1)} \geqq 0 \qquad 0 < s < \infty,\ k = 1, 2, \cdots.$$

The function $f(s) = 1/(\pi\sqrt{s})$ has this property, and the corresponding function $\alpha(t) = 2\sqrt{t}$ is clearly non-decreasing.

10. Operational considerations. One of the most fascinating aspects of the subject is its relation to the calculus of differential operators. Such considerations enable one to unify existing knowledge and to predict new results. Space is lacking for an adequate description of the method. Suffices it to say, for example, that the inversion operator for the Stieltjes transform (10) may be considered to be $-\pi^{-1}\sin(\pi s D)$, where D stands for the operation of differentiation with respect to s. If one uses the familiar exponential expression for the sine before applying the operator to $f(s)$ one is led to formula (11). On the other hand,

the infinite product expansion of the sine yields equation (12). One could use operational methods to predict inversion formulas for the higher iterates of the Laplace transform. These iterates have not yet been studied in great detail.

BIBLIOGRAPHY

1. H. S. Carslaw and J. C. Jaeger. Operational Methods in Applied Mathematics, Clarendon Press, Oxford, 1941.

2. R. V. Churchill. Modern Operational Methods in Engineering, McGraw-Hill, New York, 1944.

3. G. Doetsch. Theorie und Anwendung der Laplace-Transformation, Julius Springer, Berlin, 1937.

4. M. F. Gardner and J. L. Barnes. Transients in Linear Systems, John Wiley and Sons, New York, 1942.

5. D. V. Widder. The Laplace Transform, Princeton University Press, Princeton, 1941.

CIRCLE-TO-LINE TRANSFORMATIONS [1]

JOHN DE CICCO, Illinois Institute of Technology

1. Differential equations of the cubic type. Any ordinary differential equation of the second order which may be written in the form

$$y'' = A(x, y) + B(x, y)y' + C(x, y)y'^2 + D(x, y)y'^3, \tag{1}$$

is termed a differential equation of the cubic type. This class of differential equations has been studied extensively by Lie, R. Liouville, Tresse, Kasner, and Wilczynski. Among the important systems of curves defined as integral curves of differential equations of this type are the geodesics on any surface [2], the velocity systems of a positional field of force [3], natural families, isogonal families, Γ and Γ_0 families [4].

The class of differential equations of the cubic type (1) is transformed into itself under the group of arbitrary point transformations in the plane. In his study of the geometry of differential elements of the second order under the group of arbitrary point transformations [5], Kasner classified according to rank all differential equations of the second order which are algebraic in y' and y''. The differential equations of the first rank are exactly those of the cubic type (1).

A differential equation of the cubic type (of the first rank) may be characterized by the property that the locus of the centers of curvature of the ∞^1 integral curves passing through any point P of the plane is a general cubic curve with an isolated singularity at P, the tangent lines being the minimal lines through P. This cubical locus degenerates into a straight line only in the case of velocity systems.

2. Summary of results. In the present paper, we shall submit what is believed to be a first published proof of a theorem of Kasner which states that *the only systems of* ∞^2 *circles defined by differential equations of the cubic type* (1) *are the linear systems of circles* [6].

By this theorem and Beltrami's theorem, it can be proved that the only surfaces which can be mapped in a point-to-point fashion on a plane such that the geodesics may be represented by circles are those of constant curvature [7].

From this theorem of Kasner, we are able to obtain the complete set of point-to-point transformations T from the (x, y)-plane to the transformed (X, Y)-plane such that any straight line in the (X, Y)-plane corresponds to a circle in the (x, y)-plane. This set consists of a rational set of transformations of eleven parameters.

Some properties of these circle-to-line transformations T are obtained. Any such transformation T is rational but its inverse T^{-1} is in general irrational. The complete correspondence between the (x, y)-plane and the (X, Y) plane is two-to-one. The inverse T^{-1} is rational if and only if T is the product of a Moebius transformation by a collineation. (A Moebius transformation is a point-to-point correspondence by which every circle is converted into a circle; whereas a collineation is a point-to-point map carrying every straight line into a straight line.) In that event, the inverse T^{-1} of T is the product of a collineation by a Moebius transformation, and can be expressed as a fractional linear polygenic function in $u=x+iy$ and $v=x-iy$, with constant coefficients. Finally the only groups contained in our eleven-parameter set are the group of ∞^8 collineations and the mixed group of $2\infty^6$ Moebius transformations.

3. The statement and proof of Kasner's theorem concerning systems of ∞^2 circles of the cubic type. In this section, we shall prove the following theorem which may be considered to be an extension of the result stated in Section 2.

THEOREM 1. *A differential equation of the second order of the cubic type* (1) *can possess at most $6\infty^1$ circles as integral solutions. If it contains more than $6\infty^1$ circles as integral solutions, then every solution is a circle and the family of ∞^2 circles is linear.*

For those integral solutions of (1) which are circles, the differential equation of all ∞^3 circles: $(1+y'^2)y'''-3y'y''^2=0$, must be satisfied. Hence substituting (1) into this differential condition for circles, we obtain the differential equation of the first order and sixth degree in y' [8]

$$
\begin{aligned}
&(A_x + AB) + (A_y + B_x + B^2 + 2AC - 3A^2)y' \\
&\qquad + (B_y + C_x + A_x + 3BC + 3AD - 5AB)y'^2 \\
&\qquad + (C_y + D_x + A_y + B_x - 2B^2 - 4AC + 4BD + 2C^2)y'^3 \\
&\qquad + (D_y + B_y + C_x - 3BC - 3AD + 5CD)y'^4 \\
&\qquad + (C_y + D_x - C^2 - 2BD + 3D^2)y'^5 + (D_y - CD)y'^6 = 0.
\end{aligned} \tag{2}
$$

This equation shows that a differential equation of the cubic type (1) can possess at most $6\infty^1$ circles as integral solutions. If it has as integral solutions more than $6\infty^1$ circles, then all integral solutions are circles, and the preceding equation is an identity in y'.

From the resulting seven relations, we find that all the partial derivatives may be eliminated yielding the equations

$$(A - C)(B - D) = 0, \qquad (A - C)^2 = (B - D)^2. \tag{3}$$

From these we derive the result that $C=A$ and $D=B$. This shows that if the integral solutions of the cubic type (1) are all circles, then the differential equation (1) must represent a velocity system.

Using the conditions $C=A$, $D=B$, it is found that our seven relations reduce to the three relations

$$A_x = -AB, \qquad B_y = AB, \qquad A_y + B_x = A^2 - B^2. \tag{4}$$

By these equations, we find that there exists a function $\phi(x, y)$ such that

$$A = \phi_y, \qquad B = -\phi_x. \tag{5}$$

Replacing the A and B in (4) by (5), it follows that the function ϕ must satisfy the pair of simultaneous partial differential equations of the second order

$$\phi_{xy} = \phi_x\phi_y, \qquad \phi_{xx} - \phi_{yy} = \phi_x^2 - \phi_y^2. \tag{6}$$

First if $\phi=$const., then $A=B=C=D=0$, and the circles are the ∞^2 straight lines of the plane. Henceforth we may exclude the case where ϕ_x and ϕ_y are both zero.

It is observed that (6) may be written in the form

$$\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)\left(\frac{1}{\phi_x + i\phi_y} + \frac{x - iy}{2}\right) = 0. \tag{7}$$

Introducing the conjugate complex variables $u=x+iy$, $v=x-iy$, it evidently follows that the second parenthesis in (7) is an analytic function of u. That is

$$\phi_x + i\phi_y = \frac{2}{f(u) - v}, \tag{8}$$

where f is an analytic function of u. By taking the conjugate of the above equation and solving the result for ϕ_x and ϕ_y, we obtain

$$\begin{aligned} \phi_x &= \frac{1}{f(u) - v} + \frac{1}{g(v) - u}, \\ \phi_y &= \frac{-i}{f(u) - v} + \frac{i}{g(v) - u}, \end{aligned} \tag{9}$$

where $g(v)$ is the conjugate of the function $f(u)$.

Imposing the compatibility conditions on the function $\phi(x, y)$ whose partial derivatives are given by (9), it is seen that $f(u)$ (and hence $g(v)$) satisfies the condition [9]

$$\frac{f_u}{(f-v)^2} = \frac{g_v}{(g-u)^2}. \tag{10}$$

Consider now the case where f is constant, say $f = x_0 - iy_0$. Then (10) is satisfied and (9) becomes

$$\phi_x = \frac{-2(x-x_0)}{(x-x_0)^2+(y-y_0)^2}, \qquad \phi_y = \frac{-2(y-y_0)}{(x-x_0)^2+(y-y_0)^2}. \tag{11}$$

By (5), it follows that the differential equation of the cubic type (1) is in this case

$$y'' = -\frac{2(1+y'^2)[(y-y_0)-y'(x-x_0)]}{(x-x_0)^2+(y-y_0)^2}. \tag{12}$$

This yields all the ∞^2 circles through the fixed point (x_0, y_0), which is a linear system. Henceforth we may assume that f is not constant.

Since f is not constant, we can take the reciprocal of (10) and operate on both sides with $\partial^4/\partial u^2 \partial v^2$. The result is

$$\frac{\partial^2}{\partial u^2}\frac{1}{f_u} = \frac{\partial^2}{\partial v^2}\frac{1}{g_v}. \tag{13}$$

From this, we deduce that

$$f_u = \frac{1}{au^2+bu+c}, \tag{14}$$

where a is real and (b, c) are complex.

Next let us consider the case where a and b are both zero. Then f must be of the form $ku+l$ where $k \neq 0$, and l are complex constants. The substitution of this into (10) shows that $k = -e^{-2i\alpha}$ and $l = 2pe^{-i\alpha}$ where α and p are real numbers. Therefore by (9), we find [10]

$$\phi_x = \frac{-\cos\alpha}{x\cos\alpha + y\sin\alpha - p}, \qquad \phi_y = \frac{-\sin\alpha}{x\cos\alpha + y\sin\alpha - p}. \tag{15}$$

By (5), it is observed that the differential equation of the cubic type (1) is in this case

$$y'' = \frac{(1+y'^2)(-\sin\alpha + y'\cos\alpha)}{x\cos\alpha + y\sin\alpha - p}. \tag{16}$$

The complete solution of this yields the totality of ∞^2 circles whose centers all lie on the straight line $x\cos\alpha + y\sin\alpha - p = 0$. These form a linear system.

Henceforth we consider the case where a and b of (14) are not both zero. Substituting (14) into (10) we find

$$(17) \qquad (f - v)^2(au^2 + bu + c) = (g - u)^2(av^2 + \beta v + \gamma),$$

where β is the conjugate of b, and γ is the conjugate of c. Using the operator $\partial^2/\partial u \partial v$ on this equation and simplifying, we find

$$(18) \qquad (2au + b)f + \beta u = (2av + \beta)g + bv.$$

Hence there exists a real number d such that

$$(19) \qquad f = \frac{d - \beta u}{2au + b}.$$

Substituting (19) into (14) and simplifying, we find

$$(20) \qquad (2au + b)^2 + (b\beta + 2ad)(au^2 + bu + c) = 0.$$

If $a=0$, we find from the preceding identity in u that $b=0$. This case has been considered already. Hence $a \neq 0$, and the identity (20) in u, yields the conditions

$$(21) \qquad b\beta + 2ad + 4a = 0, \qquad b^2 - 4ac = 0.$$

Hence we find that f must be of the form

$$(22) \qquad f = -\frac{2a\beta u + b\beta + 4a}{2a(2au + b)}.$$

The last of equations (21) shows that the derivative of f with respect to u is $f_u = 4a/(2au+b)^2$. Thus the function f as given by (22) satisfies the equation (10) identically.

Substituting the value of f as given by (22) into the equations (9), simplifying and then replacing $(b+\beta)$ by $-4ax_0$, $(b-\beta)$ by $-4ay_0 i$, and $1/a$ by $k \neq 0$, we find

$$(23) \qquad \phi_x = \frac{-2(x - x_0)}{(x - x_0)^2 + (y - y_0)^2 + k}, \quad \phi_y = \frac{-2(y - y_0)}{(x - x_0)^2 + (y - y_0)^2 + k},$$

where (x_0, y_0, k) are all real numbers. The differential equation of the cubic type (1) is then

$$(24) \qquad y'' = -\frac{2(1 + y'^2)[(y - y_0) - y'(x - x_0)]}{(x - x_0)^2 + (y - y_0)^2 + k}.$$

The integral curves are the ∞^2 circles orthogonal to the real or imaginary circle $(x-x_0)^2+(y-y_0)^2+k=0$. This system is also linear.

Thus in all possible cases the only systems of ∞^2 circles which may be defined by differential equations of the cubic type (1) are those orthogonal to a fixed circle (proper with real or imaginary radius, rectilinear, or null). These form a linear system of circles. Any such system may be obtained as a linear combination of the three equations $a_j(x^2+y^2)+b_jx+c_jy+d_j=0$ for $j=1, 2, 3$, of any three non-co-axial circles. This completes the proof of Theorem 1.

4. The eleven-parameter set S_{11} of point transformations T whereby straight lines correspond to circles. We shall discuss the following result.

THEOREM 2. *If a point transformation T carries more than $6\infty^1$ circles into straight lines, then every circle of a certain linear congruence of circles is converted by T into straight lines, and T must belong to the eleven-parameter set S_{11} of transformations*

$$(25)\qquad X = \frac{a_2(x^2+y^2)+b_2x+c_2y+d_2}{a_1(x^2+y^2)+b_1x+c_1y+d_1},\qquad Y = \frac{a_3(x^2+y^2)+b_3x+c_3y+d_3}{a_1(x^2+y^2)+b_1x+c_1y+d_1},$$

where (a_j, b_j, c_j, d_j) are real numbers for $j=1, 2, 3$ whose matrix is of rank three [11].

It is noted that any transformation which does not belong to the eleven-parameter set S_{11} defined by the equations (25) can carry at most $6\infty^1$ circles into straight lines.

Let T be any transformation defined by the equations

$$(26)\qquad X = X(x, y),\qquad Y = Y(x, y),$$

where the functions (X, Y) are single valued differentiable functions of (x, y) with non-vanishing jacobian $J=\partial(X, Y)/\partial(x, y)$ in a suitable region of the (x, y)-plane.

By extending the preceding transformation twice, it is seen that the ∞^2 straight lines of the (X, Y)-plane correspond to a system of ∞^2 curves in the (x, y)-plane, which are defined by a differential equation of the cubic type (1). By Theorem 1, it follows that if T carries more than $6\infty^1$ circles of the (x, y)-plane into more than $6\infty^1$ of the totality of ∞^2 straight lines of the (X, Y)-plane, then every circle of a certain linear congruence of circles in the (x, y)-plane is converted by T into a straight line.

It may be that the linear-congruence of circles consists of the totality of ∞^2 straight lines in the (x, y)-plane. In that case T is a collineation which is included in the set (25).

Henceforth we can assume that the linear congruence of circles in the (x, y)-plane does not consist entirely of straight lines. In that event, these ∞^2 circles must be orthogonal to a fixed circle C which may be null or rectilinear or proper with a real or imaginary radius.

First of all, let us consider the case where the fixed circle C is a null-circle, that is, C is a point A. Then all the circles in the (x, y)-plane which correspond to straight lines in the (X, Y)-plane must be those passing through A.

The transformation T must be a one-to-one correspondence except for the point A. For assume the contrary and let us suppose that two distinct points p_1 and p_2, neither of which is A, in the (x, y)-plane, correspond to a single point P in the (X, Y)-plane. Then the unique circle through (p_1, p_2, A) must correspond to every straight line through P in the (X, Y)-plane. Thence every point in the

(X, Y)-plane corresponds to a point in the (x, y)-plane on the circle determined by (p_1, p_2, A). This is impossible because the jacobian J is assumed to be not identically zero. Therefore the transformation T is a one-to-one correspondence except for the point A.

If M represents an inversion with center at A in the (x, y)-plane, then the product MT^{-1} is a one-to-one correspondence converting straight lines into straight lines, and hence is a collineation P. Therefore $T=PM$, that is, T is the product of an inversion by a collineation. Any such transformation T belongs to the eleven-parameter set S_{11} defined by the equations (25).

Finally we consider the case where the fixed circle C is not a point. Then any two points in the (x, y)-plane which are inverse with respect to the circle C must correspond to a single point in the (X, Y)-plane, since the system of circles orthogonal to C is transformed into itself under an inversion with respect to C. Thus the transformation T is at least two-to-one.

We shall show that T is two-to-one. Suppose the contrary so that three distinct points (p_1, p_2, p_3) in the (x, y)-plane, where (p_1, p_2) are inverse with respect to the fixed circle C, correspond to a single point P in the (X, Y)-plane. Then every straight line in the (X, Y)-plane through the point P will correspond to the circle in the (x, y)-plane determined by the three points (p_1, p_2, p_3), which is in the linear congruence of circles since p_1 and p_2 are inverse with respect to C. Hence every point in the (X, Y)-plane corresponds to a point in the (x, y)-plane on the unique circle determined by (p_1, p_2, p_3). This is impossible since the jacobian J was assumed to be not identically zero in a suitable region of the (x, y)-plane. Therefore our transformation T is two-to-one.

This transformation T can be considered to be one-to-one by introducing the convention that two points in the (x, y)-plane shall be considered as one if and only if they are inverse with respect to the fixed circle C.

Now if S is any other transformation by which straight lines in the (X, Y)-plane correspond to circles orthogonal to the fixed circle C, then ST^{-1} is a one-to-one correspondence carrying straight lines into straight lines and hence must be a collineation P. Thus T is of the form PS, that is, T is the product of the transformation S followed by a collineation P.

Let three non-coaxial circles orthogonal to the fixed circle C be given by $a_j(x^2+y^2)+b_jx+c_jy+d_j=0$, for $j=1, 2, 3$. The matrix of the real numbers (a_j, b_j, c_j, d_j) is of rank three. Clearly one such transformation S by which all straight lines in the (X, Y)-plane correspond to circles in the (x, y)-plane must be given by equations of the type (25). Any other such transformation T is the correspondence S followed by a collineation P, which product is of the same form (25).

Thus we have shown that any transformation T by which straight lines in the (X, Y)-plane correspond to circles in the (x, y)-plane must be given by a pair of equations of the forms (25). Since any such transformation carries a certain linear congruence of circles into straight lines, our Theorem 2 is completely proved.

5. Concluding remarks. By Theorem 2, it is seen that any transformation T of our eleven-parameter set is rational. If the fixed circle C is not null, then the inverse T^{-1} is algebraic and irrational. Hence the inverse T^{-1} is rational if and only if T is either a collineation, or a Moebius transformation, or the product of a Moebius transformation by a collineation. The inverses T^{-1} of all such transformations T are given by the fractional linear polygenic functions in $u=x+iy$ $v=x-iy$ with constant complex coefficients. That is, any such rational inverse T^{-1} is given by an equation of the form

$$U = \frac{\alpha_2 u + \beta_2 v + \gamma_2}{\alpha_1 u + \beta_1 v + \gamma_1}, \tag{27}$$

where the matrix of the complex numbers $(\alpha_j, \beta_j, \gamma_j)$ for $j=1, 2$, is of rank two. All such transformations T with rational inverse form a ten-parameter subset S_{10} of S_{11}.

If one inquires as to the groups contained in the eleven-parameter set S_{11}, it is seen that if T is any transformation of S_{11} and belonging to a group G contained in S_{11}, then its inverse T^{-1} must belong to G and hence to S_{11} and therefore is of the form (27). But any transformation T^{-1} of the form (27) can belong to the subset S_{11} if and only if it is a collineation or a Moebius transformation. Hence the only groups contained in our eleven-parameter set S_{11} are the eight-parameter group of collineations and the mixed six-parameter group of Moebius circular transformations.

From the above remarks is deduced the well-known result that the group of all point transformations carrying circles into circles is the Moebius group. Also these are the only conformal transformations in our set [12].

Finally it may be stated that the jacobian J of any transformation T of the set S_{11} defined by the equations (25), vanishes only along the points of the fixed circle C defined in Theorem 2. This means that in the derivative plane, the Kasner circles corresponding to the points of the fixed circle C all pass through the origin [13].

The results of this paper are valid also in the complex cartesian plane, where any point is defined as an ordered pair of independent complex numbers.

REFERENCES

1. Presented to the American Mathematical Society, 1942.

2. Kasner, The generalized Beltrami problem concerning geodesic representation, Trans. Amer. Math. Soc., vol. 4, pp. 149–152, 1903.

3. A general velocity system is defined by the differential equation: $y''=(1+y'^2)(\psi-y'\phi)$, where (ϕ, ψ) are functions of (x, y). See Kasner, Differential geometric aspects of dynamics, Amer. Math. Soc. Publications, vol. 3, 1912, 1934. Also Trans. Amer. Math. Soc. vols., 9, 10, 1908–1909. See also Kasner and De Cicco, The geometry of velocity systems, Bull. Amer. Math. Soc., vol. 49, pp. 236–245, 1943.

4. The Γ and Γ_0 families are systems of ∞^3 curves conformally equivalent to linear families of ∞^3 circles. De Cicco, New proofs of the theorems of Beltrami and Kasner on linear families, Bull. Amer. Math. Soc., vol. 49, pp. 407–412, 1943.

5. Kasner, The geometry of differential elements of the second order with respect to the group of all point transformations, American Journal of Mathematics, vol. 28, pp. 203–213.

6. Kasner, Surfaces whose geodesics may be represented in the plane by parabolas, Trans. Amer. Math. Soc., vol. 6, pp. 141–158, 1905.

7. Beltrami, Annali di Matematica, vol. 7, 1866, pp. 185–204. See also Busse, Berliner Sitzungsberichte, 1896.

8. As an aid to deriving the equation (2), we use the equations

$$y'^2 = A^2 + 2ABy' + (B^2 + 2AC)y'^2 + 2(AD + BC)y'^3 + (C^2 + 2BD)y'^4 + 2CDy'^5 + D^2y'^6,$$

$$(2') \quad y''' = (AB + A_x) + (B^2 + 2AC + A_y + B_x)y' + (3BC + 3AD + B_y + C_x)y'^2 + (2C^2 + 4BD + C_y + D_x)y'^3 + (5CD + D_y)y'^4 + 3D^2y'^5,$$

which are obtained from (1) by squaring and differentiating.

9. In deriving the equation (10), we make use of the following operational relations

$$(10') \quad \frac{\partial}{\partial u} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial v} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right),$$
$$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}, \qquad \frac{\partial}{\partial y} = i\left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right).$$

The operators $\partial/\partial u$ and $\partial/\partial v$ are important in the theory of polygenic functions. See the last reference at end of paper.

10. As on aid to deriving the equations (15), it is noted

$$(15') \quad f(u) - v = -\varepsilon^{-2i\alpha}u + 2p\varepsilon^{-i\alpha} - v = -x(1 + \varepsilon^{-2i\alpha}) + iy(1 - \varepsilon^{-2i\alpha}) + 2p\varepsilon^{-i\alpha}$$
$$= -2\varepsilon^{-i\alpha}(x\cos\alpha + y\sin\alpha - p).$$

11. This may be contrasted with Kasner's characterization of collineations which states that if a point-to-point transformation carries more than $3\infty^1$ straight lines into straight lines, then every straight line becomes a straight line, and the transformation is a collineation. See Kasner, The characterization of collineations, Bull. Amer. Math. Soc., vol. 9, pp. 545–546, 1903.

12. For a characterization of circular transformations, see Kasner and De Cicco, Characterization of the Moebius group of circular transformations, Proc. Nat. Acad. Sci., vol. 25, pp. 209–213, 1939, and De Cicco, The analogue of the Moebius group of circular transformations in the Kasner plane, Bull. Amer. Math. Soc., vol. 45, pp. 936–943, 1939.

13. Kasner, The second derivative of a polygenic function. Trans. Amer. Math. Soc., vol. 30, pp. 803–818, 1928. Also Kasner and De Cicco, The derivative circular congruence-representation cf a polygenic function, Amer. J. Math., vol. 61, 1939, pp. 995–1003. Also a forthcoming paper in Scripta Mathematica, 1945. Finally, Kasner and DeCicco, The geometry of polygenic functions, Revista Matematicas y Fisica Teorica de la Universidad Nacional del Tucuman, Argentina, vol. 4, pp. 7–45, 1944.

Mathematics and humanism. You know how to measure the circle, you can compute the distances between the stars . . . . But if you are a real master of your profession, measure me the mind of man. Tell me how great it is or how puny.—Seneca, quoted by Cajori in *Mathematics in Liberal Education*.

. . . or on how Crane's poetry can only be defined, reviewed, and generally exposited in terms of mathematical formulae—ahem! ahem, now!—

$$\frac{\sqrt{an + pxl}}{237} = \frac{n - F_3(B^{18} + 11)}{2}.$$

—Thomas Wolfe, *You Can't Go Home Again*, p. 485.—*E. D. Schell.*

AN ELECTROMAGNETIC ANALOGY IN MECHANICS*

A. WEINSTEIN AND J. R. POUNDER, University of Toronto

1. Introduction. We propose to make a few observations on two elementary problems often treated in textbooks on classical mechanics and electromagnetic theory. We hope that our remarks will lead to a simplified and unified treatment of the subject. The problems considered are:

I. The motion of a point-charge under the influence of static uniform electric and magnetic fields.

II. The motion of a heavy particle on a rotating earth.

In the usual treatments [1], the differential equations of motion in problem I are obtained and integrated without making any approximations. The equations of motion in problem II are approximate, inasmuch as terms involving the square of ω, the earth's angular velocity, are neglected; moreover, they are usually solved only approximately, in power series. The final presentations of the solution in the two problems are not such as would suggest any connection between them. However, by writing the equations of motion in the two cases in vector form, and resolving in different directions, we shall recognize them to be identical, except for notation. We intend to give first a unified direct treatment, and then establish the link with the usual presentation.

The formulae found for the motion of the particle in problem II agree with those obtained in a paper by Denizot, who, however, did not notice the connection with the electromagnetic problem. Denizot's results, published in book form [2], were adversely criticized in the *Fortschritte der Mathematik* at the time of their appearance, and have now been completely forgotten.

2. The equations of motion. We begin with the equations of motion of the two problems:

$$m\ddot{\mathbf{r}} = e\mathbf{E} + e\dot{\mathbf{r}} \times \mathbf{H} \tag{1}$$

$$m\ddot{\mathbf{r}} = -m\mathbf{g} + m\dot{\mathbf{r}} \times 2\boldsymbol{\omega}. \tag{2}$$

Equation (1) is a statement of the classical Lorentz law of force, together with Newton's law of motion. The notation is as follows (vectors being denoted throughout by letters in heavy type):

m is the mass of the point-charge
e is its total electric charge
$\mathbf{r}$ is its position vector
$\mathbf{E}$ is the constant electric field
$\mathbf{H}$ is the constant magnetic field

Equation (2) is a statement of Newton's law of motion in a frame of reference moving with the observer. The notation is as follows:

m is the mass of the particle

* Presented to the American Mathematical Society on October 28, 1944, under the title "On Two Elementary Problems of Classical Mechanics and Electromagnetic Theory."

$\mathbf{r}$ is its position vector

$\mathbf{g}$ is the vector acceleration due to the combined gravitational and (static) centrifugal fields

$\boldsymbol{\omega}$ is the vector angular velocity of the earth (its magnitude ω is .0000729 sec.$^{-1}$)

The dot denotes differentiation with respect to the time t.

These equations are of the same form, the gravitational force corresponding to the electric force and the Coriolis force to the magnetic force. In fact, we can convert (1) into (2) by replacing e/m by 1, $\mathbf{E}$ by $-\mathbf{g}$, and $\mathbf{H}$ by $2\boldsymbol{\omega}$. That this obvious fact has not been recognized up to now is probably owing to the premature resolution of the vector equation (2) into vertical and horizontal components. Our procedure shows once more the advantage of using vectors.

We shall denote the angle btween $\mathbf{H}$ and $\mathbf{E}$ by $\pi/2-\mu$, and the angle between $\boldsymbol{\omega}$ and $-\mathbf{g}$ by $\pi/2-\lambda$, λ being the latitude.

3. The solution of the electromagnetic problem. For reasons which will become apparent later, we now proceed to solve problem I in a more explicit way than is usually presented. For the sake of definiteness we shall consider only solutions in which the particle starts from rest. We resolve equation (1) in three mutually perpendicular directions as follows (see figure 1):

1st, parallel to $\mathbf{H}$ (z-direction)

2nd, perpendicular to both $\mathbf{H}$ and $\mathbf{E}$ (y-direction)

3rd, perpendicular to both the preceding, *i.e.*, in the plane of $\mathbf{H}$ and $\mathbf{E}$ (x-direction).

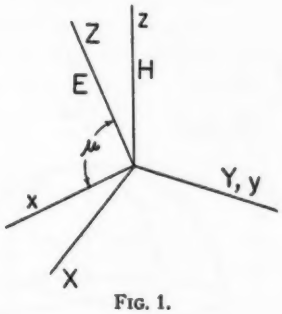


FIG. 1.

We get the following system of equations:

$$(3)\qquad \begin{cases} \ddot{x} = \dfrac{e}{m}(E\cos\mu + \dot{y}H) \\ \ddot{y} = -\dfrac{e}{m}\dot{x}H \\ \ddot{z} = \dfrac{e}{m}E\sin\mu \end{cases}$$

with the initial conditions

$$\left.\begin{aligned} x &= \dot{x} = 0 \\ y &= \dot{y} = 0 \\ z &= \dot{z} = 0 \end{aligned}\right\} \quad \text{for} \quad t = 0.$$

The solution, obtained in the usual elementary way, is:

$$(4) \qquad \begin{cases} x = \dfrac{mE\cos\mu}{eH^2}\left(1 - \cos\left(\dfrac{eH}{m}t\right)\right) \\ y = -\dfrac{E\cos\mu}{H}\left(t - \dfrac{m}{eH}\sin\left(\dfrac{eH}{m}t\right)\right) \\ z = \dfrac{eE}{m}\sin\mu\,\dfrac{t^2}{2} \end{cases}$$

We see, then, that the point-charge moves on a cycloid lying in an x, y-plane which moves parallel to itself with constant acceleration in the z-direction.

4. The solution of the mechanical problem. At this point we pass on to the solution of problem II. We resolve equation (2) in three directions corresponding to those in problem I (see figure 2):

1st, parallel to ω (z-direction).

2nd, perpendicular to both ω and $\mathbf{g}$ (y-direction).

3rd, perpendicular to both the preceding, *i.e.*, in the plane of ω and $\mathbf{g}$ (x-direction).

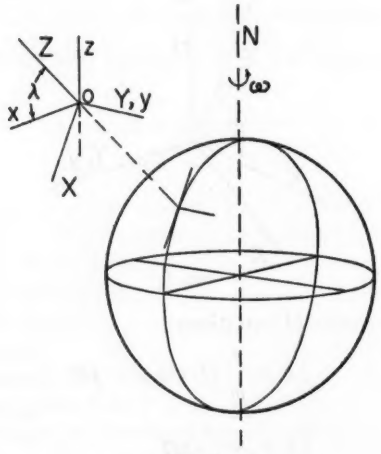


FIG. 2.

We can obtain the solution here from equations (4) by replacing H by 2ω, E by $-g$, e/m by 1, and μ by λ. We get

$$(5)\qquad \begin{cases} x = -\dfrac{g\cos\lambda}{4}(1-\cos(2\omega t)) \\ y = \dfrac{g\cos\lambda}{2}\left(t - \dfrac{1}{2\omega}\sin(2\omega t)\right) \\ z = -g\sin\lambda\,\dfrac{t^2}{2} \end{cases}$$

5. Motion in a weak magnetic field. We now wish to investigate our solution of problem I when H is small. This is of interest because of the analogy with problem II (where ω *is* small), and because the solution of I is usually given in the following form (see for instance Appell, Synge and Griffith, *l.c.* (1)):

$$(6)\qquad \begin{cases} x = A + B\cos\left(\dfrac{eH}{m}t\right) \\ y = A' - B\sin\left(\dfrac{eH}{m}t\right) - \dfrac{E\cos\mu}{H}t \\ z = C + Dt + \dfrac{eE\sin\mu}{m}\,\dfrac{t^2}{2} \end{cases}$$

where A, A', B, C, D, are constants of integration, not expressed in terms of initial conditions. A superficial inspection of equations (6) would lead to the absurd conclusion that, as H tends to zero, the velocity tends to infinity.

In order to obtain the asymptotic expressions of the solution for small values of H, we expand equations (4) as power series in $(eH/m)t$:

$$x = \frac{mE\cos\mu}{eH^2}\left(\frac{(eH)^2t^2}{m^2 2} - \frac{(eH)^4t^4}{m^4 24} + \cdots\right)$$

$$\cong \frac{e}{m}E\cos\mu\,\frac{t^2}{2} - \frac{e^3}{m^3}H^2E\cos\mu\,\frac{t^4}{24}$$

$$y = -\frac{E\cos\mu}{H}\left(\frac{(eH)^2t^3}{m^2 6} - \cdots\right)$$

$$\cong -\frac{e^2}{m^2}HE\cos\mu\,\frac{t^3}{6}$$

$$z = \frac{e}{m}E\sin\mu\,\frac{t^2}{2}.$$

We now define three directions as follows:

Z-direction: parallel to **E**

Y-direction: same as y-direction

X-direction: perpendicular to Y- and Z-directions,

i.e., we rotate the original axes about Oy through an angle $(\pi/2-\mu)$. Then

$$X \cong -\frac{e^3}{m^3} H^2 E \sin\mu \cos\mu \frac{t^4}{24}$$

$$Y \cong -\frac{e^2}{m^2} HE \cos\mu \frac{t^3}{6}$$

$$Z \cong \frac{e}{m} E \frac{t^2}{2} - \frac{e^3}{m^3} H^2 E \cos^2\mu \frac{t^4}{24}$$

so that, as H tends to zero, X and Y tend to zero, while Z tends to $(eE/m)t^2/2$, as would be expected.

6. Deviation from the vertical of a falling particle. Translating these results into the language of problem II, where the X, Y, Z, directions point south, east, and vertically upwards, we obtain the usual formulae:

$$(7) \qquad \begin{cases} X \cong \omega^2 g \sin\lambda \cos\lambda \dfrac{t^4}{6} \\ Y \cong \omega g \cos\lambda \dfrac{t^3}{3} \\ Z \cong -\dfrac{gt^2}{2} + \omega^2 g \cos^2\lambda \dfrac{t^4}{6} \end{cases}$$

7. Note. It should be noted that the formulas for the deviation of a falling particle obtained by Denizot are somewhat different from our formulae (7). This is owing to the fact that Denizot takes into account the variable part of the centrifugal force and considers instead of (2) the equation

$$\ddot{\mathbf{r}} = -\mathbf{A} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$$

where **A** is the vector representing the acceleration due to gravitation. This system of three linear equations can also be solved rigorously in an elementary way, provided **A** is considered to be constant. But this is equivalent to the usual assumption that the length of the trajectory is small compared with the radius of the earth, and in that case we are also justified in neglecting the variable part of the centrifugal force. The only difference between Denizot's formulae and ours (equations (7)) is that the numerical factor in the terms proportional to t^4 is 1/8 instead of 1/6.

REFERENCES

1. P. Appell, Mécanique Rationelle, Gauthier-Villars, Paris, 1923, Vol. I, pp. 371–376, Vol. II, pp. 284–288.

T. Levi-Civita and U. Amaldi, Meccanica Razionale, Vol. 2, Part I, Zanichelli, Bologna, 1926, pp. 141–143, 205–206.

J. L. Synge and B. A. Griffith, Principles of Mechanics, McGraw-Hill, New York, 1942, section 13.4 (esp. pp. 376–379) and section 13.5 (esp. pp. 391–395).

2. A. Denizot, Das Foucaultsche Pendel und Die Theorie der Relativen Bewegung, B. G. Teubner, Leipzig, 1913, esp. pp. 43–55.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

MINIMAL TANGENTS

IRVING KAPLANSKY, New York City

A popular problem in elementary calculus texts is to find the shortest tangent to a curve cut off between the axes, or the tangent that cuts off the triangle of smallest area, *etc.* The algebra is usually found to be fairly heavy, unlike the gratifyingly simple answers. It is perhaps interesting to solve once for all a general problem of this kind.

At the point (x, y) of a curve let the tangent be drawn; its x- and y-intercepts, say u and v, are given by

$$u = (xy' - y)/y', \qquad v = y - xy'.$$

We have $u' = yy''/(y')^2, v' = -xy''$. Suppose $z = f(u, v)$ is to be minimized. We set

$$z' = f_u u' + f_v v' = y''[f_u y/(y')^2 - f_v x]$$

equal to zero. The root $y'' = 0$ corresponds to a point of inflection which yields an uninteresting extremum. The desired solution is therefore given by

$$(y')^2 f_v/f_u = y/x. \tag{1}$$

Examples. 1. If $z = u^n + v^n$, then $f_v/f_u = v^{n-1}/u^{n-1} = (-y')^{n-1}$ and (1) becomes

$$(-y')^{n+1} = y/x. \tag{2}$$

The case $n = 2$ corresponds to minimizing the length of the tangent, the case $n = 1$ to minimizing the sum of its intercepts.

2. If $z = uv$, then $f_v/f_u = u/v = -1/y'$, yielding $-y' = y/x$. This is the problem of minimizing the area of the triangle cut off by the tangent, and fits in as the case $n = 0$ of (2). The answer has the geometrical interpretation that the triangle formed by the origin, the point of tangency, and the x-intercept is isosceles.

3. A further variant is to seek a curve for which z is constant. This is given by solving (1) as a differential equation. The solution of (2) is

$$xy = c \qquad (n = 0),$$

$$x^{n/(n+1)} + y^{n/(n+1)} = a^{n/(n+1)} \qquad (n > 0).$$

We thus establish well known properties of the rectangular hyperbola, the parabola $(n = 1)$, and the 4-cusped hypocycloid $(n = 2)$.

ON POLYNOMIALS WITH MULTIPLE ROOTS

W. J. STERNBERG, Cornell University

Let $f(x)=0$ be an algebraic equation of degree n with multiple roots. We collect roots of the same multiplicity. Let

$$\alpha_{11}, \alpha_{12}, \cdots, \alpha_{1r_1} \text{ be different roots of multiplicity } n_1,$$
$$\alpha_{21}, \alpha_{22}, \cdots, \alpha_{2r_2} \text{ be different roots of multiplicity } n_2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\alpha_{k1}, \alpha_{k2}, \cdots, \alpha_{kr_k} \text{ be different roots of multiplicity } n_k,$$

where $0<n_1<n_2< \cdots <n_k$ and $r_1n_1+r_2n_2+ \cdots +r_kn_k=n$. Put

$$(x-\alpha_{i1})(x-\alpha_{i2})\cdots(x-\alpha_{ir_i}) = g_i(x), \qquad (i = 1, 2, \cdots, k).$$

Then, assuming that the coefficient of the highest power of x is one,

$$f(x) = [g_1(x)]^{n_1}[g_2(x)]^{n_2}\cdots[g_k(x)]^{n_k}.$$

We state that the coefficients of the polynomials $g_1(x), \cdots, g_k(x)$ depend rationally on those of $f(x)$, or, to put it briefly, are "rational."

To prove this, let

$$q(x) = g_1^{n_1-1}g_2^{n_2-1}\cdots g_k^{n_k-1}$$

be the greatest common divisor of $f(x)$ and its derivative $f'(x)$. Let

$$f(x)/q(x) = h(x) = g_1g_2\cdots g_k,$$
$$f(x)/[h(x)]^{n_1} = f_1(x) = g_2^{n_2-n_1}g_3^{n_3-n_1}\cdots g_k^{n_k-n_1}.$$

The coefficients of $q(x)$, $h(x)$, $f_1(x)$ are, of course, "rational." Putting $n_2-n_1=\nu_2$, $n_3-n_1=\nu_3, \cdots, n_k-n_1=\nu_k$, we get

$$f_1(x) = g_2^{\nu_2}g_3^{\nu_3}\cdots g_k^{\nu_k}, \qquad 0<\nu_2<\nu_3<\cdots<\nu_k.$$

The polynomial $f_1(x)$ is thus of the same type as $f(x)$, but includes only $k-1$ of the polynomials $g_i(x)$. By applying to $f_1(x)$ the analogous method, we obtain

$$q_1(x) = g_2^{\nu_2-1}\cdots g_k^{\nu_k-1}, \qquad h_1(x) = g_2\cdots g_k, \qquad f_2(x) = g_2^{\nu_2-\nu_2}\cdots g_k^{\nu_k-\nu_2}.$$

The same procedure can be continued, and we conclude that the coefficients of $g_2(x), \cdots, g_k(x)$ are "rational." Moreover, $h(x)/h_1(x)=g_1(x)$. The coefficients of $g_1(x)$ are thus "rational." This completes the proof.

We mention a special case. If a root, say α_{i1} has a multiplicity n_i which is different from those of all other roots, then the corresponding polynomial $g_i(x)$ is linear, *i.e.* $g_i(x)=x-\alpha_{i1}$, and α_{i1} itself is "rational." All roots of this kind are thus "rational."

A NEW MATCH-GAME

H. D. GROSSMAN and DAVID KRAMER, New York City

Both Nim and the simple match-game: two players alternately take up to a certain number from a group of matches, the object being to take or not to take the last match, are well known. A periodical now out of print, *Games Digest*, April, 1938, suggested without solution a subtle variation of the latter, which we broaden as follows: two players alternately take up to a certain number from a group of matches, the object being to make one's opponent finish with an *odd* or an *even* total.* Analysis and generalization are simplified by focusing attention on the opponent's total (rather than one's own) and (as in Nim) on the number of matches left on the table rather than on the number taken. We shall call the players A and B. The winning moves are defined below; any other move will be called a losing one; the proof follows. A's or B's parity is the oddness or evenness of the number of matches he has at the moment.

The following solution can be worked out:

If the maximum number n one may take is	and if one's ultimate objective is to *change* his opponent's present parity, the winning move is to leave on the table a number of matches congruent to	and if one's ultimate objective is to *keep unchanged* his opponent's present parity, the winning move is to leave on the table a number of matches congruent to
I. *even*;	1 (mod $n+2$).	0 or $n+1$ (mod $n+2$).
II. *odd*;	1 or $n+1$ (mod $2n+2$).	0 or $n+2$ (mod $2n+2$).

If the move or moves indicated are impossible, you have a theoretically lost game, though you will probably win it by the subsequent failure of your opponent to make winning moves. Usually just one move is possible, except in the upper right-hand corner of the table where both moves are often possible (the latter probably preferable because it shortens the game more).

Proof. The proof will be divided into three parts:

1. A winning move can never be followed by a winning move.
2. A losing move can always be followed by a winning move.
3. The last possible winning move wins the game.

The symbols k, k' $(=k-1)$, and n represent integers. At the end of the game with an even number of matches, zero, on the table, both parities or neither will be as A (or as B) wishes. Then when the number on the table is even, both parities or neither are as A (or as B) wishes, but when the number on the table is odd, one of the parities is as A (or as B) wishes and the other is not. Naturally whatever A wishes, B always wishes the opposite.

* A special case of this game is also given in H. E. Dudeney, Amusements in Mathematics, pages 117 and 240.

I, 1. (*n even*)

A's winning move leaving	cannot be followed by B's winning move leaving	
$k(n+2)$ with B's and A's parities both as A wishes	$k'(n+2)+1$,	since B must take from 1 to n matches.
$k(n+2)+1$ with not B's but A's parity as A wishes	$k(n+2)+1$,	
$k(n+2)+n+1$ with B's but not A's parity as A wishes	$k(n+2)$ or $k(n+2)+n+1$,	

I, 2. (*n even*)

A's losing move leaving	can be followed by B's winning move leaving
$k(n+2)+2$, $k(n+2)+3$, $\cdots$, $k(n+2)+n$	$k(n+2)$ or $k(n+2)+1$, B being able to choose whichever the situation demands.
$k(n+2)$ with neither B's nor A's parity as A wishes	$k'(n+2)+n+1$.
$k(n+2)+1$ with B's but not A's parity as A wishes	$k(n+2)$ or $k'(n+2)+n+1$.
$k(n+2)+n+1$ with not B's but A's parity as A wishes	$k(n+2)+1$.

The proof of I, 3. is at once obvious. Since we start with a finite number of matches, we reach the last match in a finite number of steps. Leaving B one match changes his parity; leaving him none does not.

II, 1. (*n odd*)

A's winning move leaving	cannot be followed by B's winning move leaving	
$k(2n+2)$ with B's and A's parities both as A wishes	$k(2n+2)+1$ or $k'(2n+2)+n+1$,	since B must take from 1 to n matches
$k(2n+2)+1$ with not B's but A's parity as A wishes		
$k(2n+2)+n+1$ with neither B's nor A's parity as A wishes	$k(2n+2)$ or $k'(2n+2)+n+2$,	
$k(2n+2)+n+2$ with B's but not A's parity as A wishes		

II, 2. (*n odd*)

A's losing move leaving	can be followed by B's winning move leaving	
$k(2n+2)+2, k(2n+2)+3, \cdots, k(2n+2)+n$	$k(2n+2)$ or $k(2n+2)+1$,	B being able to choose whichever the situation demands.
$k(2n+2)+n+3, k(2n+2)+n+4, \cdots, k(2n+2)+2n+1$	$k(2n+2)+n+1$ or $k(2n+2)+n+2$,	

$k(2n+2)$ with neither B's nor A's parity as A wishes	$k'(2n+2)+n+2$.
$k(2n+2)+1$ with B's but not A's parity as A wishes	$k(2n+2)$.
$k(2n+2)+n+1$ with B's and A's parities both as A wishes	$k(2n+2)+1$.
$k(2n+2)+n+2$ with not B's but A's parity as A wishes	$k(2n+2)+n+1$.

The proof of II, 3 is the same as I, 3.

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

NATIONAL OFFICERS OF KAPPA MU EPSILON FOR 1945-46

President: E. R. Sleight, Albion College, Albion, Michigan
Vice-President: Fred W. Sparks, Texas Technological College, Lubbock, Texas
Secretary: E. Marie Hove, University of New Mexico, Alburquerque, New Mexico
Treasurer: Loyal F. Ollmann, Hofstra College, Hempstead, New York
Historian: Sister Helen Sullivan, O.S.B., Mount St. Scholastica College, Atchison, Kansas

CANTOR CENTENNIAL

Mathematics Clubs may wish to take cognizance of the fact that this year is the centennial of the birth at St. Petersburg on March 3, 1845, of the German mathematician, Georg Cantor, famous as the founder of the theory of transfinite numbers.

CLUB REPORTS 1944-45

Mathematics Club, Brown University

In each of the three semesters the Mathematics Club announced in advance its program of meetings, which were held in Wilson 307 at 2000 o'clock. Refreshments followed the formal part of each meeting. The Club picture was taken at the first meeting after Christmas. Picnics were planned for September and May. Papers for the summer semester 1944 were published in the last report. Those for the fall semester were as follows:

The design of a computing machine, illustrated by machines and lantern slides, by Professor W. Prager, Brown University, L. N. Pease presiding.

The Fibonacci series, by Evelyn Lindsay, and

The origin of logarithms, by Henry Epstein, Frances Jenckes presiding.

Ptolemy of Alexandria and his geometrical theorem, by Julianne Heller, and

The fourth dimension, by Sumner Levine, R. P. Breeding presiding.

Papers for the spring semester were as follows:

Some problems of geometry, by Professor Will Feller, Brown University, Shirley Gallup presiding.

The game of Nim, by R. B. Abel, and

One-sided surfaces, by Miriam Rose, E. N. Clarke presiding.

Magic squares, by D. T. Cross, and

Rational triangles and quadrilaterals, by P. R. Garabedian, Marjorie Hackett presiding.

A Mathematics Club Prize Competition was held during the spring semester. The prize of five dollars was awarded for the best undergraduate solution or discussion of a set of three problems in mathematics. Competitors were permitted to make full use of the Mathematical Library.

The club was directed by a Committee on Program and Arrangements, consisting of a Faculty Representative: Professor R. C. Archibald; a Student Chairman: Shirley Marilyn Gallup; and the following committee members: R. P. Breeding (fall '44), D. T. Cross (spring '45), S. L. Ehrlich, C. V. Harding, Jr., Frances Jenckes, Evelyn Lindsay, Lynn Pease.

Rho Theta, St. Louis University

Three meetings of the *Rho Theta* Mathematics Club were held in the first half of the academic year, 1944–45. A paper was read at one of them, entitled

Laplace transforms in applied mathematics, by Professor A. E. Ross.

The other two meetings were devoted to the discussion of problems arising in connection with the forthcoming affiliation with the National Honor Mathematics Fraternity, *Pi Mu Epsilon*.

The officers for 1944–45 were: President, Helen Jackson; Vice-President, Carl Kisslinger; Secretary-Treasurer, Mary F. Nawrocki.

Pi Mu Epsilon, St. Louis University

The installation of the Missouri Gamma Chapter of *Pi Mu Epsilon* at St. Louis University took place on Thursday, February 22, 1945. The program commenced with an academic procession participated in by the Arts and Science faculties. Professor P. R. Rider of Washington University in St. Louis presented the charter and delivered the principal address as the installation officer of *Pi Mu Epsilon*. The twenty-five active members of the *Rho Theta* Club became charter members of the Missouri *Gamma* Chapter and in addition forty-seven new members were initiated. The response for St. Louis University was made by the Reverend T. M. Smith, S.J., followed by a lecture on

Pathological functions, by Professor T. H. Hildebrandt of the University of Michigan.

At the banquet which culminated the day's activities, the new chapter was welcomed by the President of St. Louis University, the Reverend Patrick J. Holloran, S.J. The response for the National Organization was made by Professor Rider and for the *Rho Theta* Mathematical Fraternity, the forerunner of the new Chapter, by Miss Helen Jackson, President. The banquet speech was entitled:

The development of the mathematical sciences at St. Louis University, presented by the Reverend J. B. Macelwane, S.J., Dean of the Institute of Geophysical Technology.

At the first meeting of the Chapter, held on April 9, 1945, a paper was read, entitled:

The vector concept in elementary analytic geometry, by Brother Thomas Matthews, F.S.C.

The officers for 1944–45 are: Director, Helen Jackson; Secretary-Treasurer, Mary Nawrocki. Professor F. Regan is the Faculty Adviser and the Corresponding Secretary.

Junior Mathematics Club, Iowa State College

One meeting was held each quarter during the year, attended by an average of thirty-five persons. The following talks were given:

Navigation, by Professor D. L. Holl

The calendar, by Allen Miller and Owen Sauerlender, subsequently published under the title *The drifting years*, in *The Iowa Engineer* (vol. 45, Mar. 1945, pp. 133–135)

Matrices, by Carl Langenhop and Russell Carr.

A special invitation to the last meeting was extended to *Pi Mu Epsilon* members. The *Pi Mu Epsilon* prize in mathematics for outstanding achievements in mathematics in the freshman and sophomore years was awarded to Miss Mary Ann Williams.

The members of the committee for the year were as follows: Jean Grosser, Leslie Smith, Amanda Christensen, Adelaide Madsen, Aliene Barrett; Faculty Adviser, Professor Fred Robertson.

Mathematics Society, Massachusetts Institute of Technology

The following talks were presented during the past year:
Origins of the calculus, by Professor D. J. Struik
Continued fractions, by W. S. Loud
Fourier division, by L. R. Norwood
Cute curves, by D. L. Thomsen
Elementary topology, by O. G. Selfridge
Trisecting the angle, by R. G. Selfridge
Solution of integral equations, by R. E. Graves
The Riemann integral, by F. E. Browder
Topics in the theory of Bessel functions, by D. Mintzer
Metrics of n-dimensional space, by R. Kraichnan
Functions of a complex variable, a series of four papers, by L. R. Norwood, S. H. Crandall, R. E. Graves, and D. Mintzer
Epsilontics, by W. S. Loud
Elementary matrix theory, by I. Stempnitzky
Elementary tensor theory, by L. B. Wadel
The continuum of real numbers, by J. W. Shearer
The transcendence of e, by R. G. Selfridge
Imaginaries in geometry, by L. A. Zadeh
Diffusion, by Professor H. B. Phillips
Operational methods, by S. T. Epstein
Non-linear differential equations, by W. S. Loud
Finite geometry, by R. P. Abelson
Fourier transforms, by R. E. Graves
Probability theory, by O. G. Selfridge
Elements of Sienckiewicz theory with applications, by W. Pitts
Introductory statistics, by E. J. Gehrig
Incremental and progressional calculus, by L. B. Wadel.

In addition to the presentation of the above talks, the Society sponsored mathematics contests for Freshmen and Sophomores last fall. First places were taken by F. E. Browder and P. D. Jones, respectively. Officers of the Society were as follows. For July–October, 1944: President, L. R. Norwood; Vice-President, O. G. Selfridge; Secretary-Treasurer, S. H. Crandall. For November–June, 1944–45: President, R. E. Graves; Vice-President, L. B. Wadel; Secretary-Treasurer, R. G. Selfridge. For July–October, 1945: President, L. B. Wadel; Vice-President, F. E. Browder; Secretary-Treasurer, R. B. Davis.

Kappa Mu Epsilon, Texas Technological College

The following papers were presented during the year to the Texas Alpha Chapter of *Kappa Mu Epsilon*:

Many valued logic, by Associate Professor D. L. Webb

Life and work of Pythagoras, by Edward Turrentine

Scotopic and photopic vision, by Beverly Price

Some of the mechanical processes in making steel for guns and the construction of their barrels, by Jim Wanner

The Fourier series, its background and some original investigations, by Virginia Bowman

The fall term initiation banquet and Christmas party were combined and held in the Chimayo Room of the Hilton Hotel on December 14, 1944. Thirteen new members were initiated during the 1944–1945 session. Officers for the 1945–1946 session are: President, Jim Wanner; Vice-President, Maisie Carter; Secretary, Sarah Scroggins; Treasurer, Ben Logan; Corresponding Secretary, Lida B. May; Faculty Sponsor, Dr. D. L. Webb.

Kappa Mu Epsilon, Mount St. Scholastica College

The Kansas Gamma Chapter numbered thirty-six active members and held meetings semi-monthly. The topic for special research and study during the academic year was:

Opportunities for women trained in mathematics.

Vital information on this subject was obtained by sending form letters to fifty of the nation's leading industrial and business concerns. A symposium based on the results was presented at an all-college assembly. The participants were: Patricia Warwick, Ann Hughes, Mary Growney, Mary Davis, Virginia Harrison, Katherine Zeller, and Jane Hajovsky.

Outstanding features of the year were the following: *Mathematics and philosophy*, a lecture by the Reverend Malachy Sullivan, O.S.B.; the publication of a departmental newspaper, *The Exponent*, which was distributed to interested collegians as well as to former chapter members, and the purchase of War Bonds to promote the war effort.

At the May meeting fourteen new members were initiated, representing the states of Kansas (4), Illinois (3), Iowa (2), Alabama, Missouri, Nebraska, Pennsylvania, and Texas. Following the initiation a formal buffet supper was served. At this time the announcement was made that Katherine Zeller had merited the annual award for having contributed the most to the fraternity during the past year.

Officers elected for the year 1945–1946 are: President, Mary Lou Maloney; Vice-President, Katherine Zeller; Secretary, Mary Jane Fox; Treasurer, Mary Davis; Chapter Publicity, Victoria Fritton; Chapter Musician, Elizabeth Gulde; Corresponding Secretary and Faculty Sponsor, Sister Helen Sullivan.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Statistics and Applications. By J. G. Smith and A. J. Duncan. New York, McGraw-Hill Book Co., Inc., 1944. 10+720 pages. $4.00.

This book is a companion volume to the same authors' *Sampling Statistics and Applications*, forming a set on *Fundamentals of the Theory of Statistics*. It is designed for a beginning course and evolved over a period of ten years from mimeographed notes to its present form. Calculus is used sparingly. The contents include principles of gathering and presenting statistics, frequency distribution analysis, probability theory, normal curve, elementary sampling procedures, correlation, time series analysis, and forecasting. The more advanced points of theory are generally relegated to the companion volume. Parts of the manuscript benefited from criticism and suggestions by members of the Bureau of Labor Statistics and the Princeton mathematics department.

The book is of considerable dimensions, the introduction alone occupying 157 pages. Here are outlined applications to many fields: economics, business, politics, education, psychology, and the physical sciences. There is a description of methods of gathering statistics including discussion of types of questionnaires, illustrations being drawn from forms used by various government agencies. There is even a replica of instructions issued to census enumerators. A valuable chapter on sources goes as far back as the Domesday Book and is particularly complete as regards U. S. government publications of a statistical nature. A chapter on presentation is illustrated with six types of tables and nineteen types of charts and cartograms. The introduction ends with a discussion of certain elementary symbolic conventions and a description of graphical procedures: time series, ratio and log-log charts, discrete and continuous frequency distributions, and bivariate series.

A second division deals with the analysis of frequency distributions. Here occur descriptions of the various measures of central tendency and dispersion and their interrelations. Higher moments receive an unusually complete and lucid treatment. Change of class interval, arrangement of work sheets, and short cuts in computation are discussed in great detail and in a thoroughly practical way.

Part III, on the normal distribution, follows the classical sequence, probability—binomial distribution—normal curve. Noteworthy is an excellent critique of the Laplace and von Mises concepts of probability, leading to an acceptance of the "intuitive-axiomatic" approach in the form exposed by J. Neyman. The approach of the binomial distribution to normality is illustrated diagrammati-

cally and made acceptable with some simple analytic geometry. There is also a summary of conditions tending in practice to produce normality. Fitting a normal curve to a given histogram with subsequent χ^2 test is described. A laudable point here is the carrying of a single set of data through various aspects of a statistical analysis. Thus, data on heights of Princeton freshmen, used earlier to illustrate class interval and moments, is here fitted with a normal curve and tested. A chapter on sampling, statistical inference, and confidence limits includes a tabulation of the standard errors of the more important sampling statistics. Diagrams and clear explanations take the place of rigorous derivation.

There are 176 pages devoted to multivariate analysis, starting with simple regression and correlation and running the gamut of multiple and partial correlation into a discussion of the normal frequency surface. No pains are spared to clarify this rather confusing field. For example, detailed work sheets and diagrams and a precise attention to notation lead one through a two and a three variate analysis up to a final tour de force in four variates. Analysis of variance enters in a purely formal way in connection with multiple regression. The description of normal frequency surface includes excellent perspective figures and a detailed evaluation of the undetermined coefficients in terms of the fundamental parameters.

Part V is the study of dynamic variability from index numbers to business cycles. Notable is a paragraph on adjustment of indexes to bench marks with presentation of a method used by the Bureau of Labor Statistics. Time series are analyzed successively for trend, seasonal variation, and cyclical fluctuation. For use in higher order trends orthogonal polynomials are given a very practical discussion. Of course, orthogonality is here defined in terms of equally spaced ordinates and not over a continuous interval. Calculation of index of seasonal variation and determination of major and minor cycles after prior removal of primary factors from the raw series are described with tables and graphs. A final relatively short division deals decriptively with forecasts, a subject which the authors judiciously term an art rather than a science. Historical analogy and cross-cut analysis are the main ingredients.

There is both a subject and author index and an appendix with eight mathematical tables. Throughout the book there are 159 figures, 101 tables, and footnote references to about 300 books and papers. Very few misprints or misstatements were noted, none of them serious. There are no exercises.

The book is neither written by mathematicians nor designed primarily for mathematicians so one must overlook a certain lack of mathematical elegance sometimes apparent. The book's most striking positive characteristics are its heroic comprehensiveness in content and references, its clear elementary exposition, and its richly detailed illustrative material drawn from real life.

J. L. VANDERSLICE

College Algebra and Trigonometry. By F. H. Miller. New York, John Wiley and Sons, 1945. 12+324 pages. $3.00.

The sub-title of this text is *A Basic Integrated Course*. The titles are indicative of the premises on which the book is based; namely, that certain portions of the conventional college courses in algebra and trigonometry give the student the essential background for subsequent mathematical study, and that these fundamental parts can advantageously be integrated into a single course to serve as a foundation for the study of analytic geometry and calculus.

The choice of the material included in the book is influenced by the viewpoint of the author, that the analytical aspects of the constituent subjects are deserving of emphasis. Throughout the presentation the general concepts and the analytical techniques are the goals. Unusual deftness is shown in the treatment of a number of topics in this manner.

Chapters I and II contain a discussion of number systems, a postulational approach to algebra, and a brief treatment of functions and graphs. Chapter III is concerned with the fundamentals of trigonometry, with stress being put on the general definitions of the trigonometric functions and on radian measure. The integration of algebra and trigonometry is begun in Chapter IV, on identities and conditional equations.

Some noteworthy features of the rest of the book are the following. Emphasis is put on analytical trigonometry; only one section of three pages is devoted to the logarithmic solution of triangles. The author avails himself of the particular order and manner in which he interweaves the algebra and trigonometry to use the trigonometric functions as illustrations of transcendental functions, to relate the methods of solving algebraic and trigonometric equations, to prove selected trigonometric relations by mathematical induction, to give an unusually clear and complete treatment of complex numbers, to apply the principles and operations of algebraic inequalities to trigonometric inequalities, to study inverse trigonometric functions along with inverse algebraic functions, to give the variation of algebraic functions and the variation of trigonometric functions a common setting, and to show how certain transcendental equations of high degree may be solved with the use of methods of solving rational integral equations. Chapter XIV, on exponential and logarithmic functions, contains some well chosen material on the number e, on exponential equations, and on logarithmic equations.

The impression of the reviewer is that in *College Algebra and Trigonometry* the general concepts and analytical techniques of algebra and trigonometry necessary for the study of analytic geometry and calculus are aptly chosen and worthily treated.

T. L. WADE

Engineering Mathematics. By Harry Sohon. New York, D. Van Nostrand Company, Inc., 1944. 6+278 pages. $3.50.

Professor Sohon is Assistant Professor of Electrical Engineering at the Moore School of Electrical Engineering of the University of Pennsylvania. He has attempted to collect in this volume such mathematics as "will strengthen the student in algebra and provide him with certain mathematical tools which depend on the calculus" and has written it for the advanced undergraduate in engineering, for graduate students in engineering, and for the practicing engineer.

The Chapters in order are: Interpolation Formulas, Determinants, Dimensional Analysis, Complex Numbers and Hyperbolic Functions, Algebraic Equations, Approximate Solutions of Algebraic Equations, Fourier Series, Differential Equations, Gamma Functions and Bessel Functions, Vector Algebra, Vector Calculus, Stretched String and Round Diaphragm, and Skin Effect Problems.

It seems to this reviewer that the author has attempted to embody a large amount of elementary and advanced mathematical material in a minimum amount of space. As a result some sections in the text are almost in the character of a handbook and, in a few sections, the brevity leads to lack of coherence and continuity. Such shortness of treatment in parts of the text accounts for the absence of those helpful suggestions that make for understanding and learning—as, for example, the remark that the coefficients in Newton's interpolation formula are binomial coefficients.

There are, perhaps, three schools of thought with regard to the teaching of advanced undergraduate mathematics to engineers. One school, to characterize it in the extreme, believes in instruction by the mathematician in mathematics and by the engineer in matters engineering and a very careful cleavage in that instruction. A second school of thought, again to describe it in the extreme, believes in a single classroom with combined instruction in both advanced mathematics and engineering. The third school believes in separate instruction by the mathematician and the engineer but with frequent utilization of engineering examples in mathematics and vice versa. The present text can be classified as belonging to the first school of thought for each chapter has an opening paragraph suggesting some of the engineering applications of the material contained in that chapter. Thereafter the reading material and problems for assignment are usually mathematical in statement and content.

The chapter on Fourier Series contains an interesting analysis to show that the customary use of the trapezoidal rule in approximate analysis is to be preferred on the basis of expected accuracy to the application of Simpson's rule. The chapter on Dimensional Analysis is well written and contains several illustrations to show the underlying theory and to illustrate Buckingham's Pi Theorem. The discussion of Graeffe's method is excellent and the author illustrates with simple examples the basic ideas and restricts his theory to the cases of real roots, repeated real roots, and simple complex roots, which is sufficient for engi-

neering application. If the entire text were about one-third longer because of elaboration of material and were written in the style of these three chapters, the reviewer would find this text much more meritorious.

Misprints, rather to be expected in this period of emergency, are not too numerous and do not vitiate the content. One error of commission is made throughout the chapter on Hyperbolic Functions when the $\pm$ sign is not prefixed to the square-roots and no mention is made of the range of validity of these formulas. This same mistake sometimes is found in elementary texts on trigonometry and calculus.

The choice of content for the text includes most of the topics from advanced undergraduate mathematics that the engineering student is likely to need in specializing senior courses and in many first-year graduate courses in engineering.

J. W. CELL

Alignment Charts. By Maurice Kraitchik. New York, D. Van Nostrand Company, Inc., 1944. 6+94 pages. $2.50.

The author of this little volume was for many years in close contact with Maurice d'Ocagne, who was a pioneer in developing the science of nomography. He has here presented that science with simplicity and in a readable manner. Furthermore, the mathematical presentation is adequate and generally satisfying.

As is proper, the treatment rests on the three-rowed determinant. There is an introductory chapter on determinants as a preparation for what is to come. This material and some following chapters on the construction of scales cover twenty-five pages.

On page 26 the alignment chart is introduced in its generality. Then follow chapters on the most important special types of charts; such as those with three parallel scales and those with two parallel scales. There are chapters on special charts and other matters.

There are nearly fifty charts of various kinds. These are excellently constructed (with the single exception of Figure 19, page 36) and they would well repay study on the part of the would-be nomographer. The graduations on the scales and their numbering are models of good practice. The charts cover a variety of fields with, however, a rather strong predilection for investment problems.

The book's most noticeable defect is the unorthodox use of the word "function." We read everywhere of variables "satisfying a function" and repeatedly the author uses "function" when he means "equation."

The reviewer would have liked a somewhat fuller treatment of projective transformations. However, one should not expect too much in ninety-odd pages.

The book is generally clear and careful in its explanations and it is easy to read.

L. R. FORD

Theory of Functions. By Konrad Knopp. Translated by Frederick Bagemihl, New York, Dover Publications, 1945. 8+146 pages. $1.25.

The two parts of Knopp's Funktionentheorie have been recognized for a long time as excellent summaries of the principal results of the theory of analytic functions. The topics are well chosen and systematically arranged. While the author is careful to be accurate, he is occasionally willing to make informal supplementary remarks when these help to clarify the situation for the reader. Throughout the style is as clear and readable as is consistent with the extreme brevity.

The book under review is the translation into English of Part I, Foundations (fifth German edition). This includes the main properties of single valued analytic functions, starting with the concept of analytic function and a recapitulation of elementary properties. This is followed by a section on integral theorems, one on power series and analytic continuation, and a final section on Laurent series and singularities. The translator has done his work well, and has even made some minor corrections and improvements.

The book assumes some familiarity with the theory of real numbers, including sequences, limits and series and also with the definition and differentiation of elementary functions for complex values, and with infinite series of constant complex terms. There are some, but not very many, exercises. However, any reader with the required background can gain much from this little book. The translation should be particularly valuable to the American student for purposes of supplementary reading, self-study, or review.

PHILIP FRANKLIN

Fermagoric Triangles. By Pedro Pizá. Publication No. 1 of the Polytechnic Institute of Puerto Rico, San Germán, P.R. New York, G. E. Stechert and Co. (U. S. Distributor), 1945. 155 pages. $3.00.

A triangle with sides a, b, c, is called fermagoric of degree n in case $a^n+b^n=c^n$. For $n=2, 3, 4, 5$, the sides of such a triangle may be expressed as functions of two real parameters (say) r, s, involving at most two square root extractions. Thus for these values of n, and for rational values of r, s, the corresponding fermagoric triangles are constructible with ruler and compasses. Ingenious constructions of such triangles are a feature of the book. The author contends that F. Klein's dictum concerning the impossibility of parameterization of the Fermat equation should be revised in the light of his results. Klein's statement referred of course to expression of a, b, c as *rational functions* of parameters, reducing the Fermat equation to an *identity* in these parameters.

The latter part of the book is concerned with geometric studies of the n-fermat, that is, the algebraic plane curve of degree $4n$, traced by a point always forming a fermagoric triangle of degree n with two other fixed points as vertices of the base c.

There are many excellent figures and a wealth of numerical examples in the text.

The author, while confessing himself an amateur, is nevertheless remarkably well informed on the history of the Fermat problem, and writes with an enthusiasm all too rare in these days. One happy phrase among many lingers in the memory. In reviewing existing proofs of Fermat's theorem for many special primes, Pizá chides: "But since Euclid so gracefully demonstrated in the *Elements* that the series of prime numbers is infinite, many, no matter how many, can never be all."

C. J. EVERETT

Statistics for Sociologists. By M. J. Hagood. New York, Reynal and Hitchcock Co., 1941. 934 pages. $4.00.

As the title of the book indicates, it has been written primarily for sociologists who want to learn how to use statistical methods in their field of research and be able to evaluate critically the results of statistical analysis made by others. Since most of the sociologists have little mathematical background, the treatment of the subject, as given in the book, is elementary and non-mathematical. The mathematical tools used do not go beyond freshman college algebra. Consequently, proofs and derivations of formulas are not given, but the meaning of statistical procedures and interpretation of the results are stressed throughout the book.

The discussion of statistical methods and procedures is subdivided into two logically distinct parts, descriptive statistics and inductive statistics. In descriptive statistics, as distinguished from inductive statistics, no attempt is made to draw inference as to the parameters of the population from which the sample has been taken. The purpose of descriptive statistics is merely to condense the information contained in the sample into a few "summarizing measures" or "statistics." Under the heading "descriptive statistics" a number of summarizing measures, such as proportions, rates, frequency tables, various kinds of means, several measures of dispersion, *etc.*, are discussed. The problem of statistical inference is taken up in the part on inductive statistics. The understanding of descriptive statistics requires, of course, less intellectual effort than that of inductive statistics and, therefore, such a subdivision may have some advantages from the pedagogic point of view. On the other hand, it seems to the reviewer that the treatment of descriptive statistics as an independent logical unit has some drawbacks also, since the various "summarizing measures" can be justified, in general, only on the basis of their usefulness for purposes of statistical inference. Under the heading "statistics of relationship" contingency tables, analysis of variance and covariance, partial and multiple regressions and correlations are discussed.

Throughout the book the author takes great pains to explain the meaning of the various statistical procedures and to state the conditions under which they are applicable. While as a whole the author was successful in this effort, especially in explaining the meaning of tests of significance and estimation by confidence intervals, the discussion is perhaps not everywhere entirely adequate.

In the chapter on regressions there is no method given for setting up confidence limits for partial regression coefficients, although exact procedures for this purpose are available. The reviewer also noticed a few statements which are not entirely correct. For example, the definition of an unbiased estimate as given on page 388 is not correct. The statement that the sample standard deviation, using the number of degrees of freedom in the denominator, is an unbiased estimate of the population standard deviation is also incorrect. One can only say that the sample variance is an unbiased estimate of the population variance. In spite of these shortcomings, the book as a whole is certainly interesting and useful reading for sociologists who want to learn about statistical methods. The reader will also welcome the numerous illustrations and examples taken from fields of applications of interest to sociologists.

ABRAHAM WALD

Applied Mathematics for Technical Students. By M. S. Corrington. (Rochester Technical Series.) New York and London, Harper and Brothers. 1943. 3+226 pages. $2.20.

"This textbook was written as part of a program of the Rochester Athenaeum and Mechanics Institute for developing teaching materials of a practical nature which are closely related to the actual requirements of various jobs in industry. . . . This book . . . was written for trade schools, factory training courses, or pre-engineering studies; any mechanic or engineer can use it for home study and reference" (from the preface). The table of contents reads: I. Arithmetic with Applications, II. Fundamentals of Algebra, III. Logarithms, IV. Quadratic Equations, V. Simultaneous Equations, VI. Trigonometry with Applications. The book consists of standard material treated in a standard manner. The last three chapters are good on the whole, but the first half gives the impression of being hurriedly or carelessly written. The techniques are set forth in compactly stated rules, which are usually clear, but are sometimes awkwardly put, and even ungrammatical on occasion. There are many practical shop problems with a good worldly touch. Many of the problems are presented as excellent mechanical drawings. The author lays strong emphasis on checking.

There are quite a number of annoying details, mostly in the first half of the book. Multiplication is indicated by juxtaposition from the beginning, but this is not explained until page 58. Many problems on arithmetic in Chapter I involve mensuration formulae which are only given later. The problems on page 8 involve multiplication of fractions, which is explained on page 11. Decimals are used in problems before being explained in the text. Significant figures are mentioned on page 2, but are never defined or discussed. Some of the problems worked out in the book would set bad examples for the student in the matter of significant figures retained in the answers. The definitions of *direct* and *inverse proportion* are incorrect; two quantities are said to "vary directly" if one is a monotonically increasing function of the other. Letters are used to stand for numbers long before this procedure is explained; the author's meaning might be

guessed, however, by the student from the illustrative examples. The explanation of rational and irrational numbers on pages 51 and 52 is unclear and possibly misleading. According to the definitions on page 53 neither $5+x/2$ nor x^2 is a polynomial in x. The discussion on pages 69 and 70 gives the impression that ax^2+bx+c can be factored into linear factors with integral coefficients whenever a, b, and c are integers. The discussion of complex numbers and related difficulties in working with fractional exponents is quite confusing. It would probably have been better to omit this altogether or else to make the discussion more detailed. The reference on page 115 to Hermite's theorem on the transcendentality of e is irrelevant and would be meaningless to any reader of the book. On page 123 the notion of a function is used without any previous discussion. Plotting of curves is introduced on page 124 without any discussion of coordinates; later on pages 191 and 192 the same fault is repeated in connection with the trigonometric functions of a general angle. The student could probably understand the concrete applications of coordinates by studying the examples worked out in the book. The separate discussion of "incomplete quadratic equations" on page 127 is unnecessary. The fact that the inverse trigonometric equations are many valued is not mentioned.

There are many good warnings against common errors throughout the book. It would be better for use as a refresher under the guidance of a teacher rather than for self study. The format of the book and the layout of the pages are quite attractive.

P. C. ROSENBLOOM

NEW BOOKS RECEIVED

Sampling Statistics and Applications. By J. G. Smith and A. J. Duncan. New York, McGraw-Hill Book Co., 1945. 12+498 pages. $4.00.

Spherical Trigonometry After the Cesaro Method. By J. D. H. Donnay. New York, Interscience Publishers, Inc., 1945. 11+83 pages. $1.75.

The deflation spiral. I don't know just what a spiral is. I heard when in school that the involute of an Archimedean spiral has a circular asymptote of finite diameter. It is a beautiful way to put it, but I didn't know just what it meant.—W. H. Davis, Chairman of National War Labor Board, *Chester Wright Labor Letter*, November 13, 1943.—*E. D. Schell.*

Ideal and Real. The mathematician starts a line from an imaginary point that he informs us exists theoretically without occupying any space, which is a contradiction of terms according to his human acceptation of knowledge derived from scientific experiment, if science is based on verified facts. He assumes straight lines exist, which is a necessity for his calculating; but such a line he has never made .... He begins his study in the unknown, it ends in the unknowable.—John Uri Lloyd, *Etidorhpa*, p. 271.

The category of reality belongs not to science but to religion. It arises not as an aid to intellectual analysis, but as a means of escape or deliverance from the perplexities and confusions of deceitful appearances in a disorderly world.—M. R. Cohen, *A Preface to Logic*, p. 16.—*E. D. Schell.*

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 686. *Proposed by E. D. Schell, Arlington, Virginia*

The professor writes out a series of positive terms, $\sum a_n$, on the blackboard. Because of his carelessly written addition signs, his students consider instead the product $\prod a_n$. Unfortunately the misunderstanding cannot be discovered even though he calls out the partial sum as he writes each term. Given that a_1 and a_2 are integers, find the first five terms.

E 687. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A heavy ball is gently dropped into a vase full of water, in the shape of a segment of a paraboloid of revolution. The size of the vase is given; that of the ball is such as to cause the maximum displacement of water. Prove that the ball is just completely submerged.

E 688. *Proposed by P. A. Pizá, San Juan, Puerto Rico*

Consider the right triangle ABC with sides $a=12$, $b=5$, $c=13$. ($a^2+b^2=c^2$.) Keeping the base BC fixed, displace the vertex A without altering the perimeter (*i.e.*, along an ellipse with foci B and C) until the value of $\cos A$ is reduced from 5/13 to 7/38. Show that the sides of the new triangle satisfy the relation

$$a^3 = b^3 + c^3.$$

E 689. *Proposed by Morgan Ward, California Institute of Technology*

Let A, B, C, D be four collinear points in the order written, and let P be any other point in space. Prove that the inequality

$$PA + PD \geqq PB + PC$$

holds for all positions of P if and only if $AB=CD$.

E 690. *Proposed by F. J. Duarte, Caracas, Venezuela*

If p is a prime number greater than 3, prove that the polynomial

$$(x+y)^p - x^p - y^p$$

is divisible by $x^3+2x^2y+2xy^2+y^3$. (Cf. E 614 [1944, 591].)

SOLUTIONS

Three-Line Latin Rectangles

E 650 [1944, 586–587]. *Proposed by Lloyd Dulmage, University of Manitoba*

If we first arrange n letters in a row, in a definite order, and then arrange below these letters

(a) a second row containing the same n letters so that no letter is repeated in any column, the number of possible arrays is ${}_2K_n$;

(b) a second row containing p of the letters of the first row together with $n-p$ other letters, so that no letter is repeated in any column, then the number of arrays is ${}_2K_{n,p}$;

(c) a second row containing a definite p of the letters of the first row (and $n-p$ empty spaces) so that exactly some q of these p letters do not appear below any of the p chosen letters (but the remaining $p-q$ letters appear below some $p-q$ of the p chosen letters), no letter being repeated in any column, then the number of arrays is ${}_2K_{n,p,q}$;

(d) two rows (second and third) each containing the same n letters, so that no letter is repeated in any column, then the number of arrays is ${}_3K_n$.

Show that the functions so defined satisfy the following relations:

$$(1) \qquad {}_2K_{n,p} = \sum_{r=0}^{p} (-1)^r \binom{p}{r} (n-r)!,$$

$$(2) \qquad {}_2K_{n,p,q} = \binom{p}{q}\binom{n-p}{q} {}_2K_{p,p-q},$$

$$(3) \qquad {}_2K_n = \sum_{q=0}^{\min (p,n-p)} {}_2K_{n,p,q} \cdot {}_2K_{n-p,n-p-q}$$

(for each value of p from 0 to n),

$$(4) \qquad {}_3K_n = \sum_{p=0}^{n} (-1)^p \binom{n}{p} T_p,$$

where

$$T_p = \sum_{q=0}^{\min (p,n-p)} {}_2K_{n,p,q} ({}_2K_{n-p,n-p-q})^2.$$

Editorial Note. The proposer's solution will be published next year in the *Proceedings of the Canadian Mathematical Congress*. It is a matter of personal taste, whether one prefers this explicit (though complicated) formula for ${}_3K_n$, or the symbolic one given recently by Riordan [1944, 450–452]. Both far excel the attempts by Jacob and Kerawala (cited at the end of Riordan's article).

A Square of the Form *aabc*

E 652 [1945, 42]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

In which scales of notation can a four-digit number *aabc* be the square of a two-digit number *mn*, if $c=b+1$ and $n=m+1$?

Solution by E. P. Starke, Rutgers University. If r is the radix, we have by hypothesis

$$ar^3 + ar^2 + br + c = (mr + n)^2, \qquad a \neq 0, \tag{1}$$

which, with $c=b+1$, $n=m+1$, reduces easily to $ar^2+b=m^2(r+1)+2m$, or

$$a(r - 1) = m^2 + (2m - a - b)/(r + 1). \tag{2}$$

Hence

$$(2m - a - b)/(r + 1) = I, \tag{3}$$

an integer. Since no digit exceeds r, the possible values of I are 0, 1, -1. From (2) and (3) we have

$$a(r - 1) = m^2 + I, \qquad 2m = a + b + I(r + 1).$$

If $I=1$, elimination of r gives $(m-a)^2+ab+2a+1=0$, which is impossible for positive digits.

If $I=-1$, we have the two equations

$$a(r - 1) = m^2 - 1, \qquad 2m = a + b - r - 1.$$

Since $a \leqq r-1$, the former equation gives $a^2 \leqq m^2-1$, or $a<m$, but then the latter becomes $r-b=a-2m-1$, in which the left member must be positive, the right negative. Hence this case is also impossible.

Thus we must have $I=0$, $a(r-1)=m^2$, $2m=a+b$. Since $a \leqq r-1$, the integer $r-1$ must contain a square factor; otherwise $a=r-1=m=b$, whence $c=r$, which is impossible. Hence

$$r = xy^2 + 1,$$

where $y>1$. We may take $a=x$, $m=xy$, $b=2xy-x$, $c=2xy-x+1$, $n=xy+1$. These satisfy (1) identically. Evidently all are positive; moreover, a, m, n, b, c are all less than r, the last because $0<x(y-1)^2$. Thus the necessary and sufficient condition on r is that it can be put in the form $r=xy^2+1$, $y>1$.

The first few values are as follows:

$$\begin{aligned} r &= 5, \quad 9, \quad 10, \quad 10, \quad 13, \cdots \\ m &= 2, \quad 4, \quad 3, \quad 6, \quad 6, \cdots \\ a &= 1, \quad 2, \quad 1, \quad 4, \quad 3, \cdots \\ c &= 4, \quad 7, \quad 6, \quad 9, \quad 10, \cdots \end{aligned}$$

Also solved by Irving Kaplansky, H. L. Lee, Helen K. Milleson, W. J. Robinson, E. D. Schell, and the proposer.

A Speculation in War Bonds

E 656 [1945, 95]. *Proposed by S. H. Gould, National Research Council, Ottawa*

Each member of a very large office-staff agrees to buy a fifty-dollar war bond. One of the members suggests that each bond, instead of being given to its purchaser, be disposed of by lot. He wishes to wager one dollar that he himself will suffer loss through his own suggestion. How much should be wagered against him?

Solution by Monte Dernham, San Francisco. If there are n office workers, it is evident that in $n-1$ out of n different contingencies, all equally probable, the first bond will not go to the particular member who made the suggestion; similarly, in $(n-1)^n$ contingencies out of a total of n^n he will not draw any of the n war bonds, and will win his wager. The odds in favor of his losing the wager are therefore $n^n-(n-1)^n$ to $(n-1)^n$, or W to 1 where

$$W = \left(\frac{n}{n-1}\right)^n - 1 = \left(1 - \frac{1}{n}\right)^{-n} - 1.$$

As n increases, the decreasing function W approaches $e-1$. Thus the odds invariably exceed 1.718 to 1, no matter how large the personnel. With the aid of a sufficiently accurate table of logarithms, we find from the relation

$$n\{\log n - \log (n-1)\} = \log (W+1)$$

that $W=1.725008\cdots$ for $n=203$, but $W=1.724973\cdots$ for $n=204$. Hence, for a staff of 204 or more persons, the correct wager, to the nearest cent, is \$1.72.

Also solved by D. W. Alling, D. H. Browne, N. J. Fine, Irving Kaplansky, H. D. Larsen, Bart Park, E. D. Schell, Harry Schor, E. P. Starke, and the proposer.

Three Terms of a Geometrical Progression

E 658 [1945, 95]. *Proposed by Norman Anning, University of Michigan*

Find three three-digit numbers in geometrical progression which can be derived from one another by cyclic permutation of digits.

Solution by E. P. Starke, Rutgers University. Let the three numbers be

$$x = 100a + 10b + c, \qquad y = 100b + 10c + a, \qquad z = 100c + 10a + b.$$

If y is the geometric mean, we have $y^2=xz$ which, upon combining terms and dividing through by the factor 999, becomes

$$10(b^2 - ac) = (a^2 - bc)$$

or

$$c = (10b^2 - a^2)/(10a - b). \tag{1}$$

The trivial case $a=b=c$ is neglected. Further, if a, b have a common divisor d, then $a'=a/d, b'=b/d, c'=c/d$ give a solution whenever a, b, c do, and conversely. Hence we may take a, b relatively prime and put (1) in the form

$$c = 999a^2/(10a - b) - 10b - 100a.$$

Now $10a-b$ is also relatively prime to a, so that $10a-b$ must be a factor of 999. Thus the possible values are

$$10a - b = 1, 3, 9, 27, 37,$$

giving $(a, b) = (1, 9), (1, 7), (1, 1), (3, 3), (4, 3)$. For the first two $c>10$, the next two are trivial, and $a=4$, $b=3$ gives $c=2$, a solution. The desired numbers are 432, 324, 243; also 864, 648, 486.

Also solved by D. W. Alling, Murray Barbour, D. H. Browne, Monte Dernham, S. E. Field, F. C. Hall, Robert Hoskins, V. L. Klee, Jr., H. L. Lee, Walter Penney, E. D. Schell, J. A. Tierney, and Hazel Schoonmaker Wilson.

An Extension of the Vandermonde Determinant

E 660 [1945, 95]. *Proposed by C. D. Olds, Purdue University*

Let $z_0, z_1, \cdots, z_k$ be $k+1$ different complex numbers, all contained in the circle $|z| \leqq r$. Let

$$B_{kp} = \begin{vmatrix} 1 & z_0 & z_0^2 \cdots & z_0^{k-1} & z_0^{k+p} \\ 1 & z_1 & z_1^2 \cdots & z_1^{k-1} & z_1^{k+p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & z_k & z_k^2 & z_k^{k-1} & z_k^{k+p} \end{vmatrix}.$$

Prove that

$$\left|\frac{B_{kp}}{B_{k0}}\right| \leqq \binom{k+p}{p} r^p.$$

I. *Solution by N. J. Fine, Indianapolis.* B_{k0} is the well known Vandermonde determinant, whose value is $\prod_{i>j}(z_i - z_j)$. Expand B_{kp} by elements of the last column to get

$$B_{kp} = (-1)^k \left\{ z_0^{k+p} \prod_{i>j,\ i,j\neq 0} (z_i - z_j) - z_1^{k+p} \prod_{i>j,\ i,j\neq 1} (z_i - z_j) + \cdots \right\}.$$

Thus

$$\frac{B_{kp}}{B_{k0}} = \frac{z_0^{k+p}}{\prod_{i\neq 0}(z_0 - z_i)} + \frac{z_1^{k+p}}{\prod_{i\neq 1}(z_1 - z_i)} + \cdots + \frac{z_k^{k+p}}{\prod_{i\neq k}(z_k - z_i)}.$$

Let $F(z)=\prod_{i=0}^{k}(z-z_i)$. Then

$$\frac{B_{kp}}{B_{k0}}=\sum_{i=0}^{k}\frac{z_i^{k+p}}{F'(z_i)}=\sum_{i=0}^{k}\left(\text{residue at } z=z_i \text{ of } \frac{z^{k+p}}{F(z)}\right)$$

$$=\frac{1}{2\pi i}\int_{|z|=R>r}\frac{z^{k+p}}{F(z)}dz=\frac{1}{2\pi i}\int z^{p-1}dz\Big/\prod_{i=0}^{k}\left(1-\frac{z_i}{z}\right)$$

$$=\frac{1}{2\pi i}\int z^{p-1}\prod_{i=0}^{k}\left(1+\frac{z_i}{z}+\frac{z_i^2}{z^2}+\cdots\right)dz$$

$$=\sum z_0^{\alpha_0}z_1^{\alpha_1}\cdots z_k^{\alpha_k},$$

where the sum is taken over all sets of non-negative integers α_i such that $\sum\alpha_i=p$. The desired result now follows, since $\binom{k+p}{p}$ is the coefficient of z^p in the expansion of $(1-z)^{-(k+1)}$.

II. *Solution by Irving Kaplansky, Columbia University.* It is well known that $B_{kp}=B_{k0}H_p$, where H_p is the complete homogeneous polynomial of the pth degree in $z_0, \cdots, z_k$. The number of terms in H_p is the number of ways of selecting p objects from $k+1$ with repetitions allowed, which is $\binom{k+p}{p}$. Hence

$$\left|\frac{B_{kp}}{B_{k0}}\right|\leqq\binom{k+p}{p}r^p.$$

Also solved (by induction) by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4171. *Proposed by Tibor Radó, Ohio State University*

Let $\mathbf{x}_n$, $n=0, 1, 2, \cdots$ be a sequence of vectors in euclidean three-space such that $|\mathbf{x}_n|>\delta$ for all n, where δ is a fixed positive constant, and absolute value signs designate the length of the vector involved. Prove that the relation

$$|\mathbf{x}_n|+|\mathbf{x}_0|-|\mathbf{x}_n+\mathbf{x}_0|\to 0 \quad \text{as} \quad n\to\infty$$

holds if and only if there exists a sequence of positive scalars a_n such that $|a_n\mathbf{x}_n-\mathbf{x}_0|\to 0$ as $n\to\infty$.

4172. *Proposed by R. P. Agnew, Cornell University*

Prove or disprove the following statement involving cosine series. If $a_1, a_2, a_3, \cdots$ is a sequence of real constants, and if L_n is the length of the part of the graph of the function

$$f_n(x) = a_1 \cos x + a_2 \cos 2x + \cdots + a_n \cos nx$$

lying in the interval $-\pi \leqq x \leqq \pi$ then $L_1 \leqq L_2 \leqq L_3 \leqq \cdots$.

4173. *Proposed by Herbert Robbins, U. S. Naval Academy, Annapolis*

Let $a<b$ be given numbers and let $f(t)$ be defined, continuous, non-negative, and strictly increasing for $a \leqq t \leqq b$. By the law of the mean for integrals, for every $p>0$ there will exist a unique number $a \leqq x_p \leqq b$ such that

$$f^p(x_p) = \frac{1}{b-a}\int_a^b f^p(t)dt.$$

Find $\lim_{p\to\infty} x_p$.

4174. *Proposed by Irving Kaplansky, Harvard University*

Stone has called a ring "Boolean" if all its elements satisfy the equation $x^2=x$. Show that a ring in which $x^2=\pm x$ is either Boolean or the direct sum of a Boolean ring and the Galois field of three elements.

4175. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The twelve point sphere of any tetrahedron is the locus of the points such that the sum of the squares of their distances to the vertices diminished by their powers with respect to the circumsphere, is equal to a third of the sum of the squares of the edges.

SOLUTIONS

Factorial Coefficients

4122 [1944, 290]. *Proposed by Otto Dunkel, Washington University*

Show that $\Delta^{n-r}0^n/(n-r)!$, where n and r are non-negative integers, $n \geqq r$ is a polynomial $f_r(n)$ in n of degree $2r$. The polynomial $f_r(x)$, $r \geqq 1$, has a positive integral value for all integral values of x, positive or negative, except for $x=0, 1, 2, \cdots, r-1, r$ for which values it vanishes. If r is an odd integer $\geqq 3$, $r-1$ and r are double roots.

Solution by the Proposer. From 4108 [1945, 281] we have

$$\Delta^{n-r}0^n/(n-r)! = {}_nQ_{n-r} = \sigma_r(-n+r) = \sum_{t=1}^{r} a_t^r n^{(r+t)} = f_r(n),$$

which says that $f_r(n)$ is a polynomial of degree $2r$ in n which has the factor

$n^{(r+1)}=n(n-1)\cdots(n-r)$. If r is odd $\geqq 3$ it has the factor $n^{(r+1)}(n-r)(n-r+1)$, since it was shown in 3940 [1941, 641] that $\sigma_r(n)=n^{(r+1)}P_r(n)$ where $P_r(n)$ has the factor $n(n-1)$ if r is odd $\geqq 3$. From the equality $f_r(n)={}_nQ_{n-r}$ it follows that $f_r(n)$ is a positive integer for $n\geqq r+1$; from the equality $f_r(-n)=\sigma_r(n+r)$, $f_r(-n)$ is a positive integer for $n\geqq 1$, since $\sigma_r(n+r)$ is the rth elementary symmetric function of $1, 2, \cdots, (n+r-1)$. This completes the proof.

Number Theory

4121 [1944, 290]. *Proposed by Alfred Brauer, University of North Carolina*

In generalization of results of Sylvester and Mirimanoff the following theorem was proved, H. F. Baker, *Proc. London Math. Soc.*, (2) vol. 4, 1906, pp. 131–135.

Let d and m be relatively prime positive integers and $mm'\equiv 1 \bmod d$. Denote by $(m'x)$ the least positive residue of $m'x \bmod d$. Then

$$\frac{d^{\phi(m)}-1}{m}\equiv\sum_x\frac{(m'x)}{m-x}\bmod m,$$

where x runs over the positive integers less than m and relatively prime to m.

Recently, P. Kesava Menon proved the following theorems, *Proc. Indian Academy of Science*, Sect. A, vol. 17, 1943, pp. 107–113.

Let d, m, and n be integers such that I. $dn=m-1$; II. $dn=m+1$. Then we have

$$\text{I.}\quad \frac{d^{\phi(m)}-1}{m}\equiv\frac{1}{d}\sum_x\frac{\{x/n\}}{x}\bmod m;$$

$$\text{II.}\quad \frac{d^{\phi(m)}-1}{m}\equiv\frac{1}{d}\sum_x\frac{[x/n]}{x}\bmod m;$$

respectively, where x runs over the positive integers less than m and relatively prime to m, and where $\{a\}$ denotes the smallest integer greater than or equal to a, and $[a]$ denotes the greatest integer less than or equal to a.

Prove that these results are special cases of Baker's theorem.

Solution by the Proposer. I. We have

$$A\equiv\frac{1}{d}\sum_x\frac{\left\{\frac{x}{n}\right\}}{x}\equiv\sum_x\frac{\left\{\frac{x}{n}\right\}}{dx}\pmod m,\qquad 0<x<m,\ (x,m)=1.$$

We denote the smallest negative residue of $dx \pmod m$ by y and set

$$dx=km-y. \tag{1}$$

Since $m\equiv 1 \pmod d$, we have $(d, m)=1$, hence $(dx, m)=(y, m)=1$. If x runs over a reduced system of residues $(\bmod m)$, then dx and y also run over such a system. It follows that

$$A \equiv \sum_{x} \frac{\left\{\frac{x}{n}\right\}}{m-y} \pmod{m}. \tag{2}$$

On the other hand, set

$$B \equiv \sum_{y} \frac{(m'y)}{m-y} \pmod{m}, \qquad 0 < y < m,\ (y, m) = 1.$$

Because $m \equiv 1 \pmod{d}$, we have $m' \equiv 1 \pmod{d}$, hence $(m'y) = (y)$ and

$$B \equiv \sum_{y} \frac{(y)}{m-y} \pmod{m}. \tag{3}$$

In order to prove that $A \equiv B \pmod{m}$, it is sufficient to show that

$$\left\{\frac{x}{n}\right\} = (y)$$

for every given x less than m and relatively prime to m, where y is given by (1). We set

$$\left\{\frac{x}{n}\right\} = \left\{\frac{dx}{dn}\right\} = \left\{\frac{dx}{m-1}\right\} = l.$$

Then we have

$$l(m-1) \geqq dx > (l-1)(m-1). \tag{4}$$

Since $m-1 \geqq x$, we obtain from (4)

$$d(m-1) \geqq dx > (l-1)(m-1),$$

$$d > l-1,$$

$$d \geqq l. \tag{5}$$

On the other hand, it follows from (1) that

$$km > dx > (k-1)m. \tag{6}$$

By (4) and (6) we have

$$l(m-1) \geqq dx > (k-1)m,$$

$$(l-k+1)m > l > 0,$$

$$l > k-1,$$

$$l \geqq k. \tag{7}$$

Let us now assume that

$$l \geqq k+1. \tag{8}$$

It follows from (4) that

$$(l-1)(m-1) < dx,$$
$$(l-1)nd < dx,$$

hence by (8), (6), and (5)

$$knd < dx < km = k(nd+1) = knd + k < knd + l \leqq d(kn+1).$$

This is impossible since a multiple of d cannot lie between two consecutive multiples of d. Hence (8) is not possible, and we have $l=k$ by (7).

It follows from (1) that

$$(y) \equiv y \equiv km \equiv k \pmod{d} \tag{9}$$

since $m \equiv 1 \pmod{d}$. Moreover $(y) \leqq d$, and $k=l \leqq d$ by (5), hence by (9)

$$(y) = k = l = \left\{\frac{x}{n}\right\}.$$

II. We use again (1). Here, instead of (2) and (3), we obtain

$$A \equiv \sum_{x} \frac{\left[\frac{x}{n}\right]}{m-y} \pmod{m}, \qquad 0 < x < m,\ (x, m) = 1;$$

$$B \equiv \sum_{y} \frac{(m'y)}{m-y} \equiv \sum_{y} \frac{(dy-y)}{m-y} \pmod{m}, \qquad 0 < y < m,\ (y, m) = 1.$$

If $(a, m)=1$, then

$$\frac{1}{a} + \frac{1}{m-a} \equiv \frac{m-a+a}{a(m-a)} \equiv 0 \pmod{m},$$

hence

$$\sum_{y} \frac{1}{m-y} \equiv 0 \pmod{m},$$

where y runs over the positive integers less than m and relatively prime to m. Therefore

$$A \equiv \sum_{x} \frac{\left[\frac{x}{n}\right]+1}{m-y} \pmod{m},$$

and it is sufficient to prove that for every x with $0<x<m$, and $(x, m)=1$,

$$(dy-y) = \left[\frac{x}{n}\right] + 1 = \left[\frac{dx}{dn}\right] + 1 = \left[\frac{dx}{m+1}\right] + 1.$$

We set $[dx/(m+1)]+1=l$. We then have

$$l(m+1) > dx \geqq (l-1)(m+1) \tag{10}$$

and

$$km > dx > (k-1)m. \tag{11}$$

Hence

$$km > (l-1)(m+1) = (l-1)m + l - 1,$$

$$k > l - 1,$$

$$k \geqq l. \tag{12}$$

Let us now assume that $k \geqq l+1$. Since $m+1 > x$, it follows from (10) that $d > l-1$, hence $d \geqq l$. Therefore, by (10) and (11),

$$lnd > dx > (k-1)(nd-1) \geqq l(nd-1) \geqq lnd - d = (ln-1)d.$$

This is also impossible, and we obtain $k=l$ by (12).

On the other hand, it follows from (1) that

$$y \equiv km \equiv -k \pmod{d}$$

since $m \equiv -1 \pmod{d}$. Hence

$$dy - y \equiv -y \equiv k \pmod{d},$$

$$(dy - y) = k$$

since $k \leqq d$. Therefore

$$(dy - y) = k = l = \left[\frac{dx}{m+1}\right] + 1.$$

A review of Kesava Menon's paper appeared in the *Mathematical Reviews*, vol. 5, 1944, p. 34.

Extremal Traces for Matrices

4124 [1944, 352]. *Proposed by T. W. Anderson, Jr., Princeton University*

Consider the set of n by n matrices whose entries are positive integers or zero. Let the sum of the entries of the ith row be r_i, $i=1, 2, \cdots, n$, and the sum of the entries of the jth column be c_j, $j=1, 2, \cdots, n$. For specified r_i and c_j, positive or zero integers, with

$$\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j$$

what are the minimum and maximum sums of entries in the main diagonal, *i.e.*, the minimum and maximum traces?

Solution by H. F. Tuan and P. Halmos, Princeton University. For arbitrarily given r_i and c_j (positive or zero integers, with $\sum_{i=1}^{n} r_i = \sum_{j=1}^{n} c_j$), the set of matrices satisfying the given conditions is non-empty. This can be seen easily by applying mathematical induction on the total number of rows and columns. In fact, this process would give the maximum trace at once, for we can construct a matrix with the diagonal elements equal to min (r_i, c_i).

The transformation of adding 1 to the (h, i) entry and 1 to the (j, k) entry and subtracting 1 from the (h, k) entry and 1 from the (j, i) entry leaves the sum of each row and column invariant. This transformation is "permissible" whenever no negative entry is introduced. Using this transformation let us note some properties of a matrix (a_{ij}) with the minimum trace.

First we wish to show that there can be at most one positive entry in the main diagonal. Suppose a_{ii} and a_{jj} $(i \neq j)$ were both non-zero. Then adding 1 to (i, j) and 1 to (j, i) and subtracting 1 from (i, i) and 1 from (j, j) would be permissible. This would reduce the trace by 2 leaving the row and column totals unchanged. Since then (a_{ij}) would not have had the minimum trace, the transformation was not permissible and, hence, only one entry in the main diagonal can be non-zero. If all the entries in the main diagonal are zero, then the minimum trace is zero. Suppose a_{ii} is positive; then a_{jk} $(j, k \neq i)$ is zero. For if a_{jk} were positive, we could subtract 1 from (i, i) and 1 from (j, k) add 1 to (i, k) and 1 to (j, i), thus reducing the trace. However; the trace was minimum, hence $a_{jk}=0$, $a_{ik}=c_k$ $(k \neq j)$, $a_{ji}=r_j$ $(j \neq i)$, and $a_{ii}=r_i-\sum_{j\neq i}c_j=c_i-\sum_{j\neq i}r_j$. Hence the minimum trace is the maximum of zero and the n numbers $c_i-\sum_{j\neq i}r_j=r_i-\sum_{j\neq i}c_j$ $(i=1, 2, \cdots, n)$. It is evident that among the n numbers $c_i-\sum_{i\neq j}r_j$ $(i=1, 2, \cdots, n)$ there can be at most one which is positive, and this one, if actually positive, is the minimum trace.

Now let us consider the matrix (a_{ij}) with maximum trace. Consider an element a_{ii}. If there is another non-zero element a_{ik} in the same row and another non-zero element a_{ji} in the same column then we can add 1 to (i, i) and 1 to (j, k), subtract 1 from (i, k) and 1 from (j, i) thus increasing (i, i) and the trace. But the matrix has maximum trace so this is impossible. Hence either the ith row consists entirely of zeros except a_{ii} or the ith column consists entirely of zeros except a_{ii}, and a_{ii} is the minimum of r_i and c_i. The maximum trace, therefore, is $\sum_{i=1}\min(r_i, c_i)$.

Solved also by J. B. Kelly.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to B. W. Jones, White Hall, Cornell University, Ithaca, New York.

Professor C. J. Rees of the University of Delaware has received a citation for his work in a civilian capacity with the 14th Air Force Headquarters.

Dr. H. H. Alden has been appointed to an associate professorship at the University of Wyoming.

Dr. E. E. Blanche has been appointed to the teaching staff of the Army University organized by the War Department for American veterans at Florence, Italy.

Professor L. L. Dines of the Carnegie Institute of Technology has retired with the title professor emeritus.

Assistant Professor H. W. Eves of Syracuse University has been appointed assistant professor and head of the department of mathematics at the College of Puget Sound, Tacoma, Washington.

Assistant Professor C. H. Fischer of the University of Michigan has been promoted to an associate professorship.

Professor H. M. Gehman of the University of Buffalo has been granted leave of absence to teach in the U. S. Army Study Center at Shrivenham, England. Associate Professor Harriet Montague is acting chairman of the department in his absence.

Professor Maria D. Graham of the East Carolina Teachers College, Greenville, North Carolina, has retired.

Assistant Professor Beatrice L. Hagen of Pennsylvania State College has been promoted to an associate professorship.

Dr. Margaret P. Martin of Columbia University has been appointed to an assistant professorship in biostatistics at the University of Minnesota.

Professor L. E. Mehlenbacher of Arizona State Teachers College has been appointed to an associate professorship at the University of Detroit.

Dr. W. K. Morrill of John Hopkins University has been promoted to an assistant professorship.

Professor C. C. Morris of Ohio State University has retired.

Associate Professor C. W. Munshower of Colgate University has been promoted to a professorship.

Assistant Professor E. P. Northrop of the University of Chicago has been promoted to an associate professorship.

Dr. Helen B. Owens of Pennsylvania State College has been promoted to an assistant professorship.

Associate Professor H. H. Pixley of Wayne University has been appointed assistant dean of the College of Liberal Arts.

Assistant Professor K. C. Schraut of the University of Dayton has been promoted to an associate professorship.

The following appointment to an instructorship has been announced:
University of Michigan: Dr. George Piranian.

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

MATHEMATICS IN ENGINEERING GRADUATE STUDY

The Society for the Promotion of Engineering Education has a Committee on Graduate Study under the chairmanship of Dean L. E. Grinter of the Illinois Institute of Technology. This Committee was created in 1942, and was asked to study trends in graduate education in engineering. After three years of study, the Committee recently made its report in the form of a manual which analyzes objectives and procedures in engineering graduate study and makes recommendations concerning good practices and suggestions for improvements. The statement contained in this report about the study of mathematics as a part of the graduate curriculum in engineering is quoted.

"Graduate study in engineering emphasizes the mathematical or scientific approach to the solution of technical problems. Analytical studies are far more important in graduate work than informational courses since the art of engineering is to be learned mainly in practice rather than in graduate classes. Mathematics is the primary tool of engineering but undergraduate curricula seldom include required courses in mathematics beyond the integral calculus.

"An ability to use mathematics beyond undergraduate courses in differential and integral calculus is essential for engineering graduate study. A knowledge of differential equations has a particular value in mechanics, the functions of a complex variable are much used in electrical engineering, while an understanding of statistics as applied to the interpretation of engineering data adds to the effectiveness of the investigator. In addition, the rigorous logic of the mathematical approach (premises clearly stated; conditions of sufficiency exactly defined) is excellent training for research workers in all fields.

"It is recommended, therefore, that the master's program in engineering include one or more courses in mathematics beyond integral calculus and that the doctorate in engineering should commonly be strengthened by requiring the student to complete the equivalent of a minor study in mathematics. In some institutions these courses can be taken within the framework of the college of engineering. To meet the mathematics requirement, such courses in advanced mathematics should be taught by competent mathematicians who may also be engineers. If the courses are primarily courses in mechanics, electricity, or thermodynamics, they may be of the greatest value and still not be sufficiently rigorous mathematically to meet the objective discussed. Evidently, courses in engineering analysis should be devoted to the problem of expressing physical conditions through mathematical equations. The solution of the equations is taught in mathematics courses. Either is incomplete without the other."

PREDOCTORAL FELLOWSHIPS IN THE NATURAL SCIENCES

The National Research Council announces that it is now ready to receive nominations and applications for the predoctoral fellowships in the natural (i.e. mathematical, physical, and biological) sciences which it is administering under a grant from the Rockefeller Foundation. These fellowships are intended to assist young men and women, whose graduate study has been prevented or interrupted by the war, to complete their work for the doctorate. It is hoped that these fellowships will do much to accelerate the recovery of the scientific vigor and competence of the country which is so seriously threatened by the loss of almost two graduate school generations of scientifically trained men and women.

This program will be administered by a Committee on Predoctoral Fellowships of the National Research Council whose members are Henry A. Barton, Charles W. Bray, Detlev W. Bronk, Luther P. Eisenhart, Ross G. Harrison (Chairman—National Research Council, ex officio), W. A. Noyes, Jr., and John T. Tate, chairman: Enid Hannaford, secretary.

The annual stipend will be $1200 for single persons and $1800 for married men. In general it is expected that each recipient will spend at least eleven months per year on academic work. An additional allowance up to $500 per year will be made for tuition fees. Fellowships granted to individuals who are eligible for educational support from the "G. I. Bill of Rights" will be at such stipends as to bring the total income from these two sources to that which would be received at the above rates.

Each fellow, before entering on his graduate studies, will submit for review by the Committee on Predoctoral Fellowships a schedule, approved by the dean of his graduate school, for the completion of his work for the doctorate. This schedule, as approved by the committee, will constitute an informal agreement upon the basis of which stipend payments will be made. At the discretion of the university concerned the fellowship stipend may be supplemented by university grants. All such supplementary sources of income should be made a matter of record with the committee. The progress of the fellows will be subject to periodic review by the committee which reserves the right to cancel fellowships when in their judgment satisfactory progress is not being maintained.

Prospective candidates for these fellowships are urged to apply at once even though they may be unable to undertake their graduate study in the immediate future. Information concerning these fellowships and Nomination-Application blanks are being mailed out widely to graduate schools and wartime research laboratories. They may also be obtained by writing directly to the Secretary, Committee on Predoctoral Fellowships, National Research Council, 2101 Constitution Avenue N. W., Washington 25, D. C.

NEW NROTC UNITS

The Navy has authorized the establishment of 25 new NROTC units in addition to the 27 units already located at colleges and universities. Some V-12 students with technical interests were transferred to institutions having the new units before July 1, and additional trainees will be transferred before November 1. Essentially the traditional ROTC program will be inaugurated in all the newly selected institutions about November 1, 1945.

The new units will be at Dartmouth College, Columbia University, University of Rochester, Villanova College, Princeton University, Cornell University, Pennsylvania State College, Case School of Applied Science, Miami University, Illinois Institute of Technology, Iowa State College of Agriculture and Mechanic Arts, University of Mississippi, University of Wisconsin, University of Kansas, University of Nebraska, Alabama Polytechnic Institute, Vanderbilt University, University of Idaho, Oregon State College, Purdue University, University of Illinois, University of Missouri, University of Louisville, University of Utah, Stanford University.

POSTWAR EDUCATIONAL SERVICES FOR VETERANS AND SERVICE PERSONNEL

After signing the armistice in 1918, the Army was unprepared to carry out an educational and recreational program for its personnel before embarkation for home and during the occupation of the Rhineland. Finally, during January, 1919, a plan for an educational program was worked out by military and civilian authorities. This program was designed to be given in post schools, divisional education centers, civilian universities (for qualified graduate students), and in the AEF Educational Center located at Beaune, France. Included in the organization of the Center was the well known AEF University.

The AEF University was created to serve the needs of military personnel who desired undergraduate work of college grade and to give study opportunities to graduates for whom there were inadequate accommodations at civilian universities. It was opened March 17, 1919, and continued for only two and a half months. During that time the total registration was about 13,000 men. The faculty of the AEF University comprised military personnel with previous experience as educators and civilian instructors who were recruited into the Army Educational Corps. Students were required to have the equivalent of four years of high school before they could enter the AEF University; otherwise their needs were to be served by the post and division schools.

Keeping in mind the lessons of the last war, the Army worked for over two years on educational plans for its personnel stationed in inactive theaters of operation. These plans were completed well before VE day, and an extensive educational program is now under way, especially in the European and Mediterranean theaters. Qualified professional personnel in the Army and some civilians make up the staff for the program. Two large Army university study centers have been opened, one in England near Oxford and another in France near Paris; a number of civilian staff members for these institutions were recruited from colleges and universities in this country shortly after VE day. Educational

opportunities now generally available to military personnel do not differ greatly from those to be found in large American communities. The institutional program includes literacy training, elementary, high school, technical school, and college courses, with considerable emphasis on vocational training. Advisory and vocational guidance services are provided. Included in the plans are furloughs and field trips for educational purposes to places of cultural and historical interest. Equipment of the Army technical services and foreign civilian educational facilities are being utilized. Large quantities of classroom supplies were shipped abroad months before the end of the war.

Four types of schools have been created for participants in the educational program, namely, the unit school, the technical school, civilian universities, and Army university study centers. The unit schools have been established wherever there are separate units of 1,000 men or less. Students in these centers receive general education including literacy training, elementary, high school and junior college courses. Where equipment and instructors are available, vocational courses are offered. Technical schools have been organized where it is possible to use Army Technical Service facilities and other special equipment. These schools offer highly specialized vocational courses of a kind not generally available in the unit schools. Students are selected for training in the technical schools on a quota basis from the units taking part in the program. Civilian university centers abroad are being used and the two Army university study centers have been established to train personnel whose needs and interests go beyond the level of the unit school and the technical school. Courses are being provided in the arts and sciences as well as in the professional fields.

In addition to the institutional program, the Armed Forces Institute continues to serve specialized and individual needs. Courses available in mathematics under the auspices of the USAFI were listed in the MONTHLY for May, 1945. Since the compilation of that list, the USAFI has announced the availability to service personnel of the text, *EM 327, An Introduction to Statistical Analysis*, by C. H. Richardson.

In anticipation of demobilization, the Army has established classification and counseling sections in hospitals and separation centers throughout the United States. Here the returning service men and women are given an opportunity to discuss their problems of readjustment to civilian life. Qualified military personnel have been carefully selected and thoroughly trained in counseling, and have access to a large amount of information gathered through cooperative arrangements with governmental agencies, business, industry, labor organizations, *etc.* In order that the counselors may be well informed as to special offerings in the field of higher education, colleges and universities are invited to send to separation centers and hospitals *any printed publications* regarding special offerings for service men and women. Information concerning the number of copies required for distribution to service centers and hospitals should be obtained by contacting each Service Command Headquarters. Requests for information should be addressed to the Commanding General, Attn: Officer in

Charge Separation Classification and Counseling, of the nearest Service Command, addresses of which are as follows: Headquarters, First Service Command, Army Base, Boston, Massachusetts; Second Service Command, Governors Island, New York; Third Service Command, U. S. Post Office Building, Baltimore, Maryland; Fourth Service Command, U. S. Post Office Building, Atlanta, Georgia; Fifth Service Command, Fort Hayes, Columbus, Ohio; Sixth Service Command, Civic Opera Building, 20 North Wacker Drive, Chicago, Illinois; Seventh Service Command, Federal Building, Omaha, Nebraska; Eighth Service Command, Santa Fe Building, Dallas, Texas; and the Ninth Service Command, Fort Douglas, Utah.

Veterans' Guidance Centers have been established in 50 educational institutions throughout the country, under agreements entered into by the institutions and the Veterans' Administration. Additional agreements are rapidly being made with other institutions, and it is expected that the total number of centers will reach several hundred within a few months. The purpose of these centers is to give advice and guidance to veterans who have suffered service-connected disabilities that are pensionable and constitute vocational handicaps. Disabled veterans who are to receive vocational rehabilitation under Public Law 16, 78th Congress, are sent to these centers for advice in selecting the courses they will undertake. While there, teachers, vocational experts, psychologists, and doctors interview the veterans and give them tests to determine the type of activity they should undertake in the hope of achieving complete rehabilitation. Veterans who undertake educational courses under the G. I. Bill are not required to accept guidance or direction in selecting their courses. The services of the experts in these centers are, however, also available to veterans undertaking education under this act. The Veterans' Administration urges that those planning to return to school under the G. I. Bill take advantage of this opportunity, so that they may be assured of getting the greatest benefit from their education.

In the U. S. Navy, a program of academic, vocational and orientation education and training is well established under the Bureau of Naval Personnel. This program, known as Educational Services, has two major functions, namely, to disseminate information on the background, progress, and outcomes of the war, and to provide opportunities to continue education while in the service. This program, reaching a large percentage of Naval personnel, has grown steadily and consistently during the two and one-half years of its existence.

During the present post-war period, Educational Services provides information aimed particularly at helping the serviceman to adapt himself to his present assignment and to the post-war world. Moreover, various types of study have been made available to Navy personnel. Hundreds of thousands of men have already enrolled in classes. Subjects being taught range all the way from instruction in the three R's for non-readers and non-writers to college subjects. Use is being made of many of the Navy shops for instruction in shop and prevocational subjects. Use of USAFI correspondence courses and texts for individ-

ual study continues for personnel who are on small ships or stations or who are pursuing subjects of rather special interest.

The Demobilization Division of the Bureau of Naval Personnel is serving as coordinator for all programs that relate to the demobilization and civil readjustment of Naval personnel. The Civil Readjustment Section of the Demobilization Division has the responsibility of coordinating all programs that are concerned with the readjustment of Naval personnel to civilian life. In addition to conducting the final interview before demobilization, the Civil Readjustmen Program, through its District Civil Readjustment Officers, maintains a follow-up service to all veterans of the Navy.

THE MATHEMATICAL ASSOCIATION OF AMERICA

THE MAY MEETING OF THE WISCONSIN SECTION

The thirteenth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the State Teachers College, Milwaukee, Wisconsin, on Saturday, May 5, 1945. Sessions were held in the morning and in the afternoon. Sister Mary Felice, the Chairman of the Program Committee, presided at the morning session. The afternoon session was a joint meeting with the Milwaukee Mathematics Club and the Mathematics Section of the Wisconsin Educational Association. Mr. Lester Garbe, President of the Milwaukee Mathematics Club, presided at the afternoon session.

There were forty-five in attendance, including the following twenty-two members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, Sister M. Mirabella Boehmer, F. A. Butter, Jr., K. L. Clark, H. P. Evans, E. G. Harrell, W. W. Hart, R. C. Huffer, J. F. Kenney, Lionel London, C. C. MacDuffee, Sister Mary Felice, R. E. Norris, Elli Otteson, H. P. Pettit, P. L. Trump, B. R. Ullsvik, J. I. Vass, Louise A. Wolf.

At the business meeting the following officers were elected for the coming year: Chairman, Sister Mary Felice, Mount Mary College; Program Committee, R. C. Huffer, Beloit College, H. P. Pettit, Marquette University, B. R. Ullsvik, State Teachers College, Eau Claire. It was voted that the officers of the Wisconsin Section be authorized to make whatever arrangements seem advisable with other state groups interested in the advancement of mathematical education. It was voted that the next meeting be held in May 1946, at Mount Mary College in Milwaukee, the exact date to be set by that institution.

The following papers were presented on the morning program:

1. *A construction aid for conic sections*, by Professor H. P. Pettit, Marquette University.

It was pointed out that the construction of points on a conic by means of an auxiliary line through the intersection of the axis and the directrix, and making an angle arc sin (e) with the axis, is well known. This auxiliary line is tangent

to the conic at an end of the latus rectum. The speaker explained that for a central conic the foci can be constructed easily when the standard form is known. He then took point R on the minor axis such that $OR=OF=ae$, O being the center, and determined point S on OR, with $OS=OA=a$. He explained that the line SK parallel to RA meets OA in K on the directrix, and that the line SK is the auxiliary line. Thus the tangent at the vertex, the auxiliary line, and the tangent at the end of a minor diameter (or an asymptote in case of the hyperbola) were seen to furnish guide lines for a reasonably accurate sketch of the conic.

2. *On Kiepert's configuration*, by Dr. E. G. Harrell, State Teachers College, Platteville.

The speaker employed trilinear coördinates to develop various properties of Kiepert's configuration. Certain envelopes and other loci were also discussed.

3. *A differential notation for logarithms*, by R. S. Hoar, introduced by Professor Marden.

The logarithm of a to base b was denoted by la/lb. The symbols la and lb were called "logarentials." It was suggested that the new notation affords a useful mnemonic for converting logarithms from one base to another. It was remarked that various alternative meanings might be given to a detached logarential, one of which serves as a mnemonic for differentiating logarithms. A suggestion was made on the possible usefulness of the corresponding integral notation for anti-logarithms.

4. *Irrational numbers*, by Professor C. J. Everett, University of Wisconsin.

The speaker surveyed the principal results in the theory of irrationals, including the little known but important work of W. Maier and O. Ore. He advanced the opinion that available information now seems to warrant a modern attempt at a general theory of power series with rational coefficients. Several specific problems basic to such a project were cited.

The afternoon session was devoted to a discussion of post-war mathematics in the high school. Discussion leaders presented the papers listed below..

Professor C. C. MacDuffee presented a paper which is to be printed in a subsequent issue of this MONTHLY. Mr. Frank E. Baker, President of Milwaukee State Teachers College, discussed the philosophical, psychological, and practical values of mathematics. Mr. R. G. Chamberlin, Principal of Rufus King High School in Milwaukee, dwelt upon the concern of secondary school teachers and administrators over the question as to what position mathematics will occupy in the post-war period. This speaker itemized numerous factors which in his opinion have retarded the acceptance of mathematics.

Mr. Arpad E. Elo, research physicist for the Perfex Corporation in Milwaukee, presented industry's arguments in favor of a strong mathematics program in the high schools. He stated that the revolutionary methods of production created by the war emergency will be expanded in the future. This will demand

a higher level of technical proficiency in both skilled and unskilled workers. Our war-depleted ranks of scientific and technical personnel must be replenished from the group now in or entering high school. The opinion was advanced that "hand book engineers" suffer a great disadvantage in competition with individuals who thoroughly understand and appreciate analytical methods.

An active and valuable discussion followed the remarks of these speakers.

P. L. TRUMP, *Secretary*

THE SPRING MEETING OF THE MARYLAND, DISTRICT OF COLUMBIA, VIRGINIA SECTION

The spring meeting of the Maryland, District of Columbia, Virginia Section of the Mathematical Association of America was held at George Washington University on Saturday, May 12, 1945. Professor C. H. Wheeler III, Chairman of the Section, presided at the morning and afternoon sessions.

There were forty-four persons present, including the following twenty-five members of the Association: G. R. Clements, Abraham Cohen, E. L. Crow, Alexander Dillingham, J. A. Duerksen, P. J. Federico, Michael Goldberg, D. W. Hall, M. A. Hyman, F. E. Johnston, L. M. Kells, M. H. Martin, Florence M. Mears, T. W. Moore, W. K. Morrill, R. E. Root, E. D. Schell, M. F. Smiley, C. V. L. Smith, A. D. Sollins, J. H. Taylor, C. C. Torrance, F. J. Weyl, C. H. Wheeler III, R. H. Wilson, Jr.

At the business meeting the following officers were elected for the coming year: Chairman, J. B. Scarborough, United States Naval Academy; Secretary-Treasurer, E. J. Finan, Catholic University of America; Members of the Executive Committee, W. K. Morrill, Johns Hopkins University, D. W. Hall, University of Maryland; Regional Governor, Gillie A. Larew, Randolph-Macon Woman's College.

The morning program consisted of the following papers:

1. *Equations of motion of classical dynamical systems of variable mass*, by Lieut. Joseph Giarratana, U. S. Naval Academy.

The speaker derived a set of general equations of motion for classical dynamical systems of variable mass. The variation of mass considered was due to either a continuous deformation and motion of the defining surface, or to a motion of the material points of the system, or to both. Special cases of the general equations were also considered.

2. *Stresses in rectangular plates*, by Lieut. C. B. Lindquist, U. S. Naval Academy.

The problem considered was that of an isotropic rectangular plate stressed in its own plane uniformly by a tensile stress on one edge, and held in static equilibrium by uniform shear forces on the two adjacent edges. The stress function F (from which the stresses were obtained by differentiation) was expressed in terms of an infinite series constructed to satisfy the biharmonic equation

$\nabla^4 F = 0$ and the boundary conditions on F and its normal derivatives. The conditions of the normal derivatives lead to an infinite system of linear equations in infinitely many unknowns, the unknowns being the undetermined constants appearing as coefficients in the series for the stress function F. This system of equations was shown to have a unique, bounded solution. A similar method was used to solve the problem of material having two directions of elastic symmetry parallel to the edges of the plate.

The afternoon speaker was Professor Tobias Dantzig of the University of Maryland, who gave an interesting talk on *The Bequest of the Greeks*.

W. K. MORRILL, *Secretary*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at Hamline University in St. Paul, Minnesota, on Saturday, May 12, 1945. Sessions were held in the forenoon, at noon, at luncheon, and in the afternoon. Professors K. H. Bracewell, E. J. Camp, Dean W. H. Bussey, and Professor L. E. Bush presided at the respective sessions.

Thirty-five persons attended the meetings, including the following nineteen members of the Association: Walter Bartky, R. W. Brink, L. E. Bush, W. H. Bussey, E. J. Camp, C. S. Carlson, S. Elizabeth Carlson, Brother Louis De La Salle, I. C. Fischer, Gladys Gibbens, W. L. Hart, C. M. Jensen, J. M. H. Olmstead, Abraham Spitzbart, F. J. Taylor, Marian W. Thornton, Ella Thorp, K. W. Wegner, G. L. Winkelmann.

The following officers were elected for the coming year: Chairman, C. S. Carlson, St. Olaf College; Secretary, L. E. Bush, College of St. Thomas; Executive Committee, J. M. H. Olmsted, University of Minnesota, K. W. Wegner, Carleton College, Brother Louis De La Salle, Saint Mary's College.

On a motion by Professor Brink, a résumé of the career of Professor Anthony L. Underhill of the University of Minnesota, who died on January 18, 1945, was entered on the minutes of the meeting.

At the noon session Dean Walter Bartky of the University of Chicago delivered a lecture on *Linear Systems of Differential Equations*. Dean Bartky applied matrices to the solution of the system

$$\frac{dx_i}{dt} = \sum_{j=1}^{n} a_j x_j, \qquad (i = 1, 2, \cdots, n)$$

and showed how the labor of solving certain systems of this type which arise in industry could be greatly reduced.

In addition to the lecture by Dean Bartky, the following papers were presented:

1. *On Stieltjes' integral equations*, by Mr. Monroe D. Donsker, University of Minnesota.

The speaker reported on a paper appearing in the *Annals of Mathematics* in 1922 concerning the application of the classical Fredholm theory of integral equations to Stieltjes' integral equations of the type

$$u(x) = f(x) + \lambda \int_a^b u(y) d_y K(x, y).$$

2. *Hyperbolic inversion*, by Miss Esther Reizman, University of Minnesota, introduced by Dean Bussey.

The inverses of a point and of a curve with respect to a circle were defined in the usual way, and then defined again in terms of poles and polars. Generalized inversion with respect to any central conic was then defined on the basis of this second definition of circular inversion. Inversion with respect to the hyperbola $xy=1$ received special attention. Some facts about hyperbolic inverses of the general conic were presented. Some special cases and illustrations were described.

3. *An application of the theory of envelopes to a problem in light*, by Professor E. J. Camp, Macalester College.

It was the speaker's purpose to call attention to a class of problems from elementary physics which are applications of the theory of envelopes, but which are usually ignored in texts on differential equations.

Consider a family of parallel plane waves given by the equation $y=kx+b_n(t)$. Suppose that the y-axis is the cross section of a reflecting mirror, and that α_1 the inclination of the lines, is an obtuse angle. Then $\alpha_1-90°$ is the angle of incidence. Points where the incident waves strike the y-axis are to be regarded as centers of circular waves whose equations can be written in the form

$$x^2 + (y - \beta)^2 = \frac{[b_n(t) - \beta]^2}{k^2 + 1}.$$

The envelope of this family of circles for a fixed t and a fixed n is a straight line with equation $y=-kx+b_n(t)$. Upon comparing this equation with the first equation in this paragraph, it is seen that the angle of incidence is equal to the angle of reflection. Other problems which yield to a similar analytic treatment can be obtained by considering incident waves with circular wave fronts.

4. *Matrices and quadric surfaces*, by Professor J. M. H. Olmsted, University of Minnesota.

Invariants of real quadratic equations in three variables are easily obtained by the use of matrices for distance preserving transformations, by means of the artifice of a fourth variable equal to 1. Professor Olmsted proved that although the characteristic roots of the fourth order matrix of a real quadratic form in three variables are not invariant, their signs are. This fact extends to higher dimensional spaces, and leads to a general characterization theorem for quadric hypersurfaces, involving the characteristic roots of the associated matrices of orders n and $n+1$.

5. *Analyzing objectives in the teaching of mathematics*, by Brother Louis De La Salle, Saint Mary's College.

The speaker discussed the advantages for the instructor and for the student of having a very explicit statement of objectives. He advocated a statement specific enough to enable the individual student to recognize his attainment. Excerpts from such a statement were distributed by way of illustration.

6. *Bernard Bolzano*, by Sister M. Thomas a Kempis, College of Saint Teresa.

In anticipation of the centenary of the death of Bernard Bolzano in 1848, the speaker presented a short biography which stressed his success in freeing mathematical analysis from intuitional treatment.

7. *The ancient quadratrix from a modern point of view*, by Dean W. H. Bussey, University of Minnesota.

The quadratrix was invented by Hippias (c. 425 B.C.) for the purpose of angle trisection, and was later used by Deinostratus (c. 350 B.C.) for the quadrature of the circle. Its equation may be written $x = y \cot(\pi y/2)$. The small part of the curve which is described in books on the history of elementary mathematics is a curved line from the point (0, 1) to (1/2, 1/2), and on to meet the x-axis at $(2/\pi, 0)$. When this curve and its equation were explained to a class in analytic geometry, one of the students inquired regarding the appearance of the rest of the curve. Dean Bussey gave a short summary of what he learned about the quadratrix in answering that question.

L. E. BUSH, *Secretary*

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Annual Meeting, Chicago, Ill., November 24–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

You will want to read

FERMAGORIC TRIANGLES

By PEDRO A. PIZÁ

(A publication of the Polytechnic Institute of Puerto Rico)

An attractive piece of work which has impressed me favorably. L. CARLITZ

A book which marks a new and important departure in this field of mathematics. ALEXANDER KLEMIN

Find it very interesting. Congratulations. W. B. CARVER

Using truly elementary mathematics you seem to have blown a new life into the famous Fermat problem. The word "fermagoric" delights me as a perfect alloy. HOWARD EVES

Contains new and brilliant ideas. J. GINSBURG

I have started reading "Fermagoric Triangles" and find many interesting features. EDWARD KASNER

This unusual book is an interesting piece of work, carefully executed and written with refreshing enthusiasm. "Fermagoric" triangles have not been systematically discussed hitherto. My all good wishes for success. H. S. M. COXETER

Order now from U. S. distributor: Cloth bound copies $3.00 postpaid.

G. E. STECHERT & CO.
31 East 10th St., New York, N.Y.

BOOK NEWS

ANALYTIC GEOMETRY, Revised Edition

By Raymond W. Brink

FOR a rich, complete, and adaptable course in analytic geometry, this is an ideal text. Throughout the emphasis is on logic and method. Problems, carefully graded and distinctive for their variety, freshness, and usefulness, provide for immediate application of theory, and formal exercises give training in the development of technique. The book is particularly useful as preparation for calculus and other advanced mathematical studies. *8vo, 350 pages.* $2.90.

ESSENTIALS OF ANALYTIC GEOMETRY

By Raymond W. Brink

SIMPLER, more flexible, and somewhat less detailed than the author's *Analytic Geometry*, this book covers fully all topics usually taught in analytic geometry courses in American colleges and universities. It is compact, well-organized, and clearly written, and it provides sufficient material in method, information, and problems for preparation for calculus and for the development of general mathematical maturity and background. *Small 8vo, 233 pages.* $2.40.

D. APPLETON-CENTURY COMPANY
35 West 32nd Street New York 1, New York

Back Numbers Wanted

For a limited number of copies, forty cents a copy will be paid (in stamps for amounts less than one dollar) for any of the following issues of the AMERICAN MATHEMATICAL MONTHLY:

1910, May
1913, Apr., May, June, Oct.
1914, Jan., Apr.
1928, Nov.
1929, Aug.-Sept.
1930, Jan.
1931, Feb., Oct. (Part I only), Dec.
1940, Jan.
1941, Jan., Apr., May
1942, May

Anyone having for sale volumes 1 to 11 inclusive, or *odd* copies of these volumes, will please communicate with the undersigned.

W. B. CARVER, *Secretary-Treasurer*
MATHEMATICAL ASSOCIATION OF AMERICA
McGraw Hall, Cornell University
ITHACA, NEW YORK

A COMPLETE REFERENCE BOOK

for college students in trigonometry through calculus

THE JAMES MATHEMATICS DICTIONARY

provides:

the facts the student has learned in presupposed subjects, the forgetting of which causes most of his current difficulties;

correlation between his various subjects by means of its carefully worked out cross-reference system;

tables—logarithmic; trigonometric; differentiation; extensive integral; denominate numbers; mathematical symbols; squares and cubes, and mathematics of finance.

A life-saver for students who have been out of school for some time.

Attractive format, durable fabricoid binding, either flexible or non-flexible, price $3.00. Fifteen percent discount to teachers. Larger discounts on quantity orders.

THE DIGEST PRESS, Dept. 1A
VAN NUYS CALIFORNIA

UNCLASSIFIED

Calculus

by Sherwood and Taylor

● The outstanding feature of this text is its integration of differential and integral calculus. The authors have designed a text to promote **understanding,** not merely rote learning.
Fundamental concepts, such as limits, functions, definite integrals, and convergence of sequences are fully explained. Definite integrals are discussed **before** indefinite integrals. The treatment of infinite series and their use is more complete than is usual in calculus texts. There are many examples, exercises, and problems.

503 pages **College List $3.75**

The Gist of Mathematics

by Moore and Mira

● In this imaginative text for freshman survey courses the development and application of mathematics is traced. Stress is placed on coherence of the various mathematical fields. The authors take nothing for granted in the mathematical preparation of the students. The book is thorough and elementary; still it is painstakingly scholarly.

Answers—College List $1.00 **726 pages** **College List $3.75**
Tables..College List .30

Wartime Refresher in Fundamental Mathematics

by Eddy, Pulliam, Brolly, Upton, and Thomas

● Ideal for a concentrated review of mathematical essentials, this refresher consists of twenty one-hour lessons, each embracing one topic. A concise glossary of mathematical rules and definitions is appended. The text is used without outside references.

248 pages **College List $1.05**

College Mathematics: A First Course

by W. W. Elliott and E. Roy C. Miles

● The essentials of college algebra and plane trigonometry, and an introduction to plane analytic geometry and differential and integral calculus are presented in this one volume mathematics course. It is a basic text for a full-year freshman course.

396 pages **College List $3.00**

College Algebra

With Revisions and Additional Problems

by Harold Thayer Davis

● A good deal of attention is given to the historical incidents which marked the development of algebra, and a chapter on "recreational mathematics" has been included to make the subject more interesting to the student. There are approximately 3,000 problems.

470 pages **College List $2.85**

for approval copies address

PRENTICE-HALL, INC. 70 FIFTH AVENUE, NEW YORK 11

Standard Macmillan Texts

COMPUTATION AND TRIGONOMETRY

By Harold J. Gay

Beginning with the general angle, this book gives a more detailed discussion of computation than is ordinarily included in trigonometry texts. Starting with Chapter IV, the book presents the material generally covered in a text on plane and spherical trigonometry. *With tables, $2.50. Without tables, $2.00*

A SHORT COURSE IN TRIGONOMETRY

By James G. Hardy

This work differs significantly from the usual text in that numerical solution of triangles by logarithmic methods has been relegated from the customary first place in importance to a secondary position.

2nd Edition. With tables, $2.25. Without tables, $1.75

CALCULUS

By Robbins & Little

In this text topics usually presented in an introductory chapter are taken up throughout the book as the need arises and each new principle is presented by means of a specific problem. Special attention has been paid to the exercises of which there are nearly 2300. *$3.25*

A FIRST COURSE IN DIFFERENTIAL EQUATIONS

By Nelson B. Conkwright

Improvement in the exposition of tried material rather than the introduction of novel features has been the author's aim in writing this text. The material is developed with applied problems revealing the lines of investigation in which differential equations can be put to practical use. *$2.25*

SURVEY OF MODERN ALGEBRA

By Birkhoff & MacLane

Presupposing nothing but high school algebra, this text gives an integrated study of the theory of equations, elementary number theory, and modern algebra, with emphasis on the developments in modern algebra which have exerted influence on modern physics, philosophy, and higher mathematics. *$3.75*

The Macmillan Company · 60 Fifth Avenue · New York 11

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN

V528
C745
UMI

9

TECHNOLOGY DEPT.

LIBRARY
DEC 1 2 1945

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 52 NUMBER 9

CONTENTS

Elementary Evaluation of Laplace Transforms	D. E. RICHMOND	481
Limits for the Field of Values of a Matrix	A. B. FARNELL	488
A Circle Covering Theorem	A. W. GOODMAN, R. E. GOODMAN	494
he Lagrange Identity as a Unifying Principle	LOUIS BRAND	499
Magic Rectangles Modulo p	E. J. FINAN	502
Discussions and Notes	J. M. H. OLMSTED, VICTOR THÉBAULT	507
Clubs and Allied Activities		510
Recent Publications		514
Problems and Solutions		516
News and Notices		533
General Information		536
The Cooperative Committee on Science Teaching		536
A Course for Prospective Teachers in Ohio		536
Academic Interests of Discharged Veterans		537
Teachers for The Veteran Rehabilitation Program		538
Important Legislation Before Congress		539
The Mathematical Association of America		541
New Members		541
Calendar of Future Meetings		542

NOVEMBER 1945

The AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF THE

MATHEMATICAL ASSOCIATION OF AMERICA, INC.

THIS MONTHLY WAS FOUNDED IN 1894 BY BENJAMIN F. FINKEL

LESTER R. FORD, Editor-in-Chief

WITH THE CO-OPERATION OF

E. F. BECKENBACH
E. T. BELL
L. M. BLUMENTHAL
N. B. CONKWRIGHT
H. S. M. COXETER
OTTO DUNKEL
H. P. EVANS
HOWARD EVES
B. F. FINKEL
J. S. FRAME
ORRIN FRINK, JR.
MARJORIE GROVES
B. W. JONES
C. V. NEWSOM
HAIM REINGOLD
MARIE J. WEISS

EDITORIAL CORRESPONDENCE should be addressed to the EDITOR-IN-CHIEF, L. R. FORD, Illinois Institute of Technology, Technology Center, Chicago 16, Ill.

BOOKS FOR REVIEW should be addressed to REVIEW EDITOR, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association, as well as correspondence regarding the business of the Association, should be sent to the SECRETARY-TREASURER, W. B. CARVER, McGraw Hall, Cornell University, Ithaca, N. Y. Change of address must reach the Secretary-Treasurer before the 10th of the month in which the change becomes effective.

OFFICERS OF THE ASSOCIATION

EXECUTIVE COMMITTEE: President, C. C. MACDUFFEE, Wisconsin; First Vice-President, W. M. WHYBURN, Texas Technological College; Second Vice-President, W. F. CHENEY, JR., Connecticut; Secretary-Treasurer, W. B. CARVER, Cornell; Editor-in-Chief, L. R. FORD, Illinois Institute of Technology. Associate Secretary, B. W. JONES, Cornell. Honorary President, W. D. CAIRNS, Oberlin.

Additional Members of the BOARD OF GOVERNORS: WALTER BARTKY, R. W. BRINK, H. S. M. COXETER, N. A. COURT, L. L. DINES, R. C. HUFFER, GILLIE A. LAREW, J. W. LASLEY, JR., SAUNDERS MAC LANE, H. F. MAC NEISH, W. L. MISER, E. J. MOULTON, A. L. NELSON, F. S. NOWLAN, E. J. PURCELL, W. R. RANSOM, R. G. SANGER, G. W. SMITH, H. L. SMITH, MORGAN WARD.

Entered as second class matter at the post office at Menasha, Wis. Acceptance for mailing at special rate of postage provided for in the Act of February 28, 1925, embodied in Paragraph 4, Section 538, P. L. and R., authorized April 1, 1926.

SUBSCRIPTION PRICE: To Members, $4 a Year, To Others, $5 a Year.

PUBLISHED BY THE ASSOCIATION MENASHA, WIS., AND CHICAGO, ILL.

UMI

ELEMENTARY EVALUATION OF LAPLACE TRANSFORMS

D. E. RICHMOND, Williams College and Institute for Advanced Study

1. Introduction and evaluation of simple transforms. It is well known that Laplace transforms are increasingly applied to the solution of engineering problems, notably in the study of transients in filters and transmission lines. From a pedagogical point of view it is therefore of interest to observe that it is possible to build up a very extensive table of transforms by means within the reach of students familiar with the solution of elementary differential equations. It is even possible to go a considerable distance without performing a single integration. It is believed that the presentation of these methods may prove useful to teachers of undergraduate courses in mathematics.

A function $f(s)$ is said to be the Laplace transform of $F(t)$ if $f(s)=\int_0^\infty e^{-st}F(t)dt$. Hereafter this relationship between $f(s)$ and $F(t)$ will be abbreviated $f(s)=L\{F(t)\}$. The following theorems are well known and easily derived under suitable restrictions on $F(t)$.

If $f(s)=L\{F(t)\}$,

I. $sf(s)-F(0)=L\{F'(t)\}$,

II. $f'(s)=L\{-tF(t)\}$,

III. $f(s-a)=L\{e^{at}F(t)\}$.

Moreover, if $f_1(s)=L\{F_1(t)\}$ and $f_2(s)=L\{F_2(t)\}$,

IV. $c_1f_1(s)+c_2f_2(s)=L\{c_1F_1(t)+c_2F_2(t)\}$, where c_1 and c_2 are arbitrary constants. In particular, it follows that $0=L\{0\}$.

It is proposed in the following to make almost exclusive use of I–IV. In only one case will it be necessary to appeal to the integral representation. The student should be warned, however, that I and II fail to apply in the case of certain functions $F(t)$. He is perfectly safe if $F'(t)$ is continuous for $t>0$, $tF(t)\to 0$ as $t\to 0$ and $|F'(t)|<ce^{at}$ for some c and a, and for all t sufficiently large.

In the present section we shall develop some of the more elementary transforms from I–IV. We shall first find the Laplace transform of 1. If $x(s)=L\{1\}$, then by I, $sx(s)-1=L\{0\}=0$. Hence $x(s)=1/s$.

$$\frac{1}{s}=L\{1\}. \tag{1}$$

We have tacitly assumed that a given function of t, in this case 0, has only one transform, which of course is true.

In similar fashion, $1/s^2=L\{t\}$, and by induction

$$\frac{1}{s^{n+1}}=L\left\{\frac{t^n}{n!}\right\} \quad \text{for } n \text{ a positive integer.} \tag{2}$$

$$\frac{1}{s-a}=L\{e^{at}\}, \quad \text{using (1) and III.} \tag{3}$$

Similarly, if

$$x(s) = L\{\sin at\},$$
$$sx(s) = L\{a \cos at\},$$
$$s^2x(s) - a = L\{-a^2 \sin at\},$$

Hence

$$s^2x(s) - a = -a^2x(s), \qquad x(s) = \frac{a}{s^2 + a^2}$$

$$\frac{a}{s^2 + a^2} = L\{\sin at\}, \quad \text{and by } I, \tag{4}$$

$$\frac{s}{s^2 + a^2} = L\{\cos at\}. \tag{5}$$

From these results, one obtains by III,

$$\frac{1}{(s-a)^{n+1}} = L\left\{\frac{t^n e^{at}}{n!}\right\}, \tag{6}$$

$$\frac{a}{(s-b)^2 + a^2} = L\{e^{bt} \sin at\}, \tag{7}$$

$$\frac{s-b}{(s-b)^2 + a^2} = L\{e^{bt} \cos at\}; \tag{8}$$

and by II,

$$\frac{2as}{(s^2 + a^2)^2} = L\{t \sin at\}, \tag{9}$$

$$\frac{s^2 - a^2}{(s^2 + a^2)^2} = L\{t \cos at\}. \tag{10}$$

The only explicit use of the integral representation which we shall need occurs in the case of the transform of t^α ($\alpha > -1$). We obtain $L\{t^\alpha\} = \Gamma(\alpha+1)/s^{\alpha+1}$ by substituting $x = st$ in $\int_0^\infty e^{-x}t^\alpha dt$. It will be useful to recall that $\Gamma(1/2) = \sqrt{\pi}$ and $\Gamma(3/2) = \sqrt{\pi}/2$. The method of this section is similar to the treatment of Churchill.*

2. More difficult transforms. To develop the less elementary transforms it is useful to prepare the following table† by successive use of I and II:

* Churchill, R. V. Modern Operational Mathematics in Engineering, McGraw-Hill, 1944.

† Equivalent to that given by van der Pol, B., On the operational solution of linear differential equations and an investigation of the properties of these solutions, Philosophical Magazine (7), vol. 8, 1929, p. 877. Except for the use of the asymptotic theorems, our method is the same as his.

$f(s)$	$F(t)$	
x	X	
$sx - X_0$	X'	$X_0 = X(0)$
$s^2x - sX_0 - X_1$	X''	$X_1 = X'(0)$
$-x'$	tX	
$-sx' - x$	tX'	
$-s^2x' - 2sx + X_0$	tX''	
x''	t^2X	
$sx'' + 2x'$	t^2X'	
$s^2x'' + 4sx' + 2x$	t^2X''	

Sufficient conditions for the validity of any entry are readily stated in consequence of those for Theorems I and II.

We shall also need the following asymptotic theorems:*

V. If $X(t) \sim At^\alpha$ $(\alpha > -1)$ as $t \to 0$,

$$x(s) \sim \frac{A\Gamma(\alpha+1)}{s^{\alpha+1}} \quad \text{as} \quad s \to \infty.$$

VI. If $X(t) \sim Bt^\beta$ $(\beta > -1)$ as $t \to \infty$,

$$x(s) \sim \frac{B\Gamma(\beta+1)}{s^{\beta+1}} \quad \text{as} \quad s \to 0.$$

By way of explanation of the notation,

$$F(t) \sim G(t) \text{ is equivalent to } \frac{F(t)}{G(t)} \to 1.$$

Example 1. To find the transform of $J_0(t)$. $J_0(t)$ is a solution of $tX'' + X' + tX = 0$. Using the table, we get at once the transformed equation:

$$s^2x' + x' + sx = 0.$$

By a simple integration, $x(s) = c/\sqrt{s^2+1}$. To determine c, note that $X(t) \sim 1$ as $t \to 0$. Hence $x(s) \sim 1/s$ as $s \to \infty$. But by inspection $x(s) \sim c/s$. Hence $c = 1$ and

$$\frac{1}{\sqrt{s^2+1}} = L\{J_0(t)\}. \tag{11}$$

Using $J_0'(t) = -J_1(t)$ and I,

* Doetsch, G. Theorie und Anwendung der Laplace-Transformation, Dover, 1943, pp. 188, 200.

$$\frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}} = L\{J_1(t)\}. \tag{12}$$

Example 2. Let $X(t)=\sin a\sqrt{t}$. By differentiating twice and eliminating, one readily obtains

$$4tX'' + 2X' + a^2X = 0.$$

Using $X_0=0$ and the table,

$$4s^2x' + (6s - a^2)x = 0, \quad \text{which gives}$$

$$x(s) = \frac{c}{s^{3/2}} e^{-a^2/4s}.$$

Now $X(t)\sim at^{1/2}$ as $t\to 0$. Hence $x(s)\sim a\sqrt{\pi}/2s^{3/2}$ as $s\to\infty$. By inspection $x(s)\sim c/s^{3/2}$, hence $c=a\sqrt{\pi}/2$ and

$$\frac{a\sqrt{\pi}}{2s^{3/2}} e^{-a^2/4s} = L\{\sin a\sqrt{t}\}, \quad \text{and by I,} \tag{13}$$

$$\frac{\sqrt{\pi}}{s^{1/2}} e^{-a^2/4s} = L\left\{\frac{\cos a\sqrt{t}}{\sqrt{t}}\right\}. \tag{14}$$

Example 3. $X(t)=J_0(a\sqrt{t})$ satisfies

$$4tX'' + 4X' + a^2X = 0.$$

The transformed equation is

$$4s^2x' + (4s - a^2)x = 0, \quad \text{which gives at once}$$

$$x(s) = \frac{ce^{-a^2/4s}}{s}.$$

Now $X(t)\sim 1$ as $t\to 0$, and $x(s)\sim 1/s$ as $s\to\infty$. Thus $c=1$ and

$$\frac{e^{-a^2/4s}}{s} = L\{J_0(a\sqrt{t})\}. \tag{15}$$

3. Another method. It is often easier to reverse this procedure, seeking the $X(t)$ which corresponds to a given $x(s)$. It will be assumed that $x(s)$ satisfies a linear differential equation with polynomial coefficients. The method consists in making such a linear combination of entries from the left side of the following table as to give this differential equation. The same linear combination of the right-hand entries gives the associated differential equation for $X(t)$, which may be found by integration and application of the asymptotic theorems.

$f(s)$	$F(t)$
x	X
x'	$-tX$
x''	t^2X
$sx - X_0$	X'
sx'	$-tX' - X$
sx''	$t^2X' + 2tX$
$s^2x - sX_0 - X_1$	X''
$s^2x' + X_0$	$-tX'' - 2X'$
s^2x''	$t^2X'' + 4tX' + 2X$

If it is necessary to use lines 4, 7, or 8, involving X_0 (and X_1), these quantities should be calculated by use of the known relations

$$X_0 = \lim_{s\to\infty} sx(s) \quad \text{and} \quad X_1 = \lim_{s\to\infty} [s^2x(s) - sX_0].$$

If line 7 is to be used, X_0 and X_1 must be 0. If line 4 or 8 occurs separately, X_0 must be 0; if both are used, sx and s^2x' must have equal coefficients. In case these conditions are not satisfied, simple modifications of the method may be made. (See example 6.)

Inverse transforms found by this method should be checked by the method of §2 to assure the validity of the results.

Example 4. Consider $x(s) = e^{-a\sqrt{s}}/\sqrt{s}$. By differentiation and elimination,

$$4sx'' + 6x' - a^2x = 0.$$

From the table,

$$4t^2X' + 2tX - a^2X = 0$$

and $X(t) = c/\sqrt{t}\, e^{-a^2/4t}$. Now $X(t) \sim c/\sqrt{t}$ as $t \to \infty$. Hence by VI,

$$x(s) \sim \frac{c\Gamma(1/2)}{\sqrt{s}} = \frac{c\sqrt{\pi}}{\sqrt{s}} \quad \text{as} \quad s \to 0.$$

By inspection, $x(s) \sim 1/\sqrt{s}$ so $c = 1/\sqrt{\pi}$ and

$$\frac{e^{-a\sqrt{s}}}{\sqrt{s}} = L\left\{\frac{1}{\sqrt{\pi t}}\, e^{-a^2/4t}\right\}. \tag{16}$$

The functions of t are such as to justify the argument by reversing the steps. The order chosen makes it necessary to solve only the simpler of the associated differential equations.

Example 5. $$x(s) = \frac{(\sqrt{s^2+1} - s)^n}{\sqrt{s^2+1}}$$

is suggested by (12). One easily obtains

$$(s^2 + 1)x'' + 3sx' + (1 - n^2)x = 0.$$

From the table,

$$t^2X'' + tX' + (t^2 - n^2)X = 0,$$

Bessel's equation of nth order.

$$\text{Now } x(s) = \frac{1}{\sqrt{s^2+1}(\sqrt{s^2+1} + s)^n} \sim \frac{1}{2^n s^{n+1}} \text{ as } s \to \infty.$$

We should choose $X(t)$ such that $X(t) \sim t^n/2^n n!$ as $t \to 0$. Thus $X(t) = J_n(t)$.

Example 6. $x(s) = e^{-1/s}/s^{1/2}$. Then $2s^2x' + (s-2)x = 0$. To get the equation for $X(t)$, we shall need to use lines 4 and 8 of the table, involving X_0. But $X_0 = \lim_{s\to\infty} s^{1/2}e^{-1/s} = \infty$, and the method fails. We use instead of $x(s)$, the function $e^{-1/s}/s^{3/2}$. The method will then work, giving

$$\frac{e^{-1/s}}{s^{3/2}} = L\left\{\frac{\sin 2\sqrt{t}}{\sqrt{\pi}}\right\} \quad cf.\ (13).$$

The answer to the original problem is now easily found as in (14).

It will be found that almost all the entries in tables of transforms in such standard texts as Churchill and Doetsch may be found by the use of this method or that of §2.

4. An important example.* A more difficult problem is to solve for $X(t)$

$$x(s) = \frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}} = L\{X(t)\}.$$

It is easy to show that $X_0 = 1$.

By repeated differentiation and a fairly difficult elimination, one obtains

$$(s^2 + 1)x''' - (3s^2 - 5s + 3)x'' + (2s^2 - 10s + 7)x' + (2s - 5)x = 0.$$

Adding to the table of §3 the entries

$$x''' = L\{-t^3X\} \quad \text{and} \quad s^3x''' = L\{-t^3X'' - 6t^2X' - 6tX\},$$

one gets the equation

$$(17) \qquad (t + 1)(t^2 + 2t)X'' + (t^2 + 2t + 2)X' + (t + 1)^2X = 0.$$

Setting $u = t^2 + 2t$,

* A considerable simplification of this section was made as the result of a suggestion by the referee.

$$4u\frac{d^2X}{du^2}+4\frac{dX}{du}+X=0.$$

Comparing with Example 3, it will be seen that $J_0(\sqrt{u})=J_0(\sqrt{t^2+2t})$ is the solution of (17) for which $X_0=1$. Consequently

$$\frac{e^{s-\sqrt{s^2+1}}}{\sqrt{s^2+1}}=L\{J_0(\sqrt{t^2+2t})\}. \tag{18}$$

A related result is widely used in the theory of transmission lines:

$$x(s)=\frac{e^{-\sqrt{s^2+1}}}{\sqrt{s^2+1}}=L\left\{\begin{matrix}0,\ t<1;\\ J_0(\sqrt{t^2-1}),\ t\geqq 1\end{matrix}\right\}. \tag{19}$$

This result follows from (18) by the use of a well-known shift theorem, according to which

$$e^{-as}f(s)=L\left\{\begin{matrix}0,\ t<a;\\ F(t-a),\ t\geqq a\end{matrix}\right\}$$

if $f(s)=L\{F(t)\}$. If our method is applied directly to (19), it is found that $x(s)\sim e^{-s}/s$ as $s\to\infty$. The asymptotic theorems do not apply immediately, but the result obtained suggests multiplication of $x(s)$ by e^s to bring the problem within the scope of our method. A similar device is applicable to a number of other known cases.

It is possible that the method of §3 may lead to the discovery of new transforms. Thus

$$x(s)=\frac{e^{\tan^{-1}s}}{\sqrt{1+s^2}}$$

yields $(1+s^2)x'+(s-1)x=0$, the transformed equation of $tX''+X'+(1+t)X=0$. We desire the solution with $X_0=e^{\pi/2}$. This solution may be written

$$X(t)=e^{\pi/2}[1-t+t^2/9+\cdots]$$

but seems not to be expressible in closed form.

Probability in Literature. Through chance, we are each a ghost to all the others, and our only reality; through chance, the huge hinge of the world, and a grain of dust; the stone that starts an avalanche, the pebble whose concentric circles widen across the seas.—Thomas Wolfe, *Look Homeward Angel.*

Three times he dropped a shot so close to the boat that the men at the oars must have been wet by the splashes . . . each shot deserved to be a hit he knew, but the incalculable residuum of variables in powder and ball and gun made it a matter of chance just where the ball fell in a circle of fifty yards radius, however well aimed.—C. S. Forester, *Captain Horatio Hornblower.*—*E. D. Schell.*

LIMITS FOR THE FIELD OF VALUES OF A MATRIX

A. B. FARNELL, West Point, N. Y.

1. Introduction. Let $A=(a_{rs})$ be a square matrix of order n with complex numbers as elements. The totality of numbers $\sum_{r,s}a_{rs}\bar{x}_r x_s$, where $\sum_r x_r\bar{x}_r$ has the value unity, comprise the *field of values* of the matrix A. The equation $|\lambda I-A|=0$ is called the *characteristic equation* fo the matrix A, and the roots λ_r, the *characteristic roots* of the matrix A. The maximum of the $|\lambda_r|$ is called the *dominant characteristic root.*

In 1900, Bendixson [3] obtained upper limits for the real and imaginary parts of the characteristic roots of a real matrix. In 1902, Hirsch [9] extended these results to matrices with complex numbers as elements. A limit was also given by Bromwich [4] in 1904. These limits were further refined by Browne [5] in 1930, and by Parker [10] in 1937.

In 1918, Toeplitz [11], using the results of Bendixson and Hirsch, showed that the field of values lies within a rectangle with sides parallel to the real and imaginary axis. Hausdorff [8] showed that the field of values is connected, bounded, closed, and convex.

If λ is a characteristic root of a matrix A of order n, there exist a set of numbers $(x_1, x_2, \cdots, x_n)$, such that

$$\sum_{r=1}^{n} x_r\bar{x}_r = 1,$$

which satisfy the relations

$$\lambda x_r = \sum_{s=1}^{n} a_{rs}x_s \qquad (r = 1, 2, \cdots, n). \tag{1}$$

If we multiply the rth equation in (1) by $\bar{x}_r$, and sum as to r, we obtain

$$\lambda = \sum_{r,s=1}^{n} a_{rs}\bar{x}_r x_s. \tag{2}$$

Hence the characteristic roots lie within the field of values of A. In the following, λ will be considered to be any value in this field, the conclusions holding a fortiori where λ is a characteristic root.

In a previous paper [6] the following theorem was established.

THEOREM 0. *If λ lies in the field of values of A,*

$$|\lambda| \leqq \left(\sum_{r,s}|a_{rs}|^2\right)^{1/2},$$

i.e., the absolute value of λ is not greater than the square root of the sum of the squares of the absolute values of the elements of A.

2. Some new limits. Let $\xi_r = |x_r|$. Then, using (2),

$$|\lambda| \leqq \sum_{r,s} |a_{rs}| \xi_r \xi_s = \sum_{r,s} d_{rs} \xi_r \xi_s,$$

where

$$d_{rs} = \tfrac{1}{2}(|a_{rs}| + |a_{sr}|), \qquad (d_{rs}) = D.$$

In view of this, the first two theorems obtained previously [6] may be restated as follows:

THEOREM 1. *Let* $S_r = \sum_s d_{rs}$, $S = \max(S_r)$. *Then, if* λ *lies in the field of values of* A,

$$|\lambda| \leqq S.$$

THEOREM 2. *If* λ *lies in the field of values of* A, *then,*

$$|\lambda| \leqq \left(\sum_{r,s} d_{rs}^2 \right)^{1/2}.$$

3. Observations on these limits. Let $\overline{A}$ denote the conjugate of A, and write

$$B = (A + \overline{A}')/2, \qquad C = (A - \overline{A}')/2i.$$

Then, if $\lambda = \alpha + i\beta$, it can easily be shown that

$$\alpha = \sum_{r,s} b_{rs} \bar{x}_r x_s, \qquad \beta = \sum_{r,s} c_{rs} \bar{x}_r x_s,$$

i.e., α and β satisfy inequalities similar to those for λ.

It is interesting to note that two well-known results follow perhaps most easily from any theorem of the type given above. If A is Hermitian, C equals zero, and therefore the field of values lies on the real axis, and the characteristic roots are real. And if A is skew-Hermitian, *i.e.*, if $A + \overline{A}'$ equals zero, the field of values lies on the imaginary axis, and the characteristic roots are pure imaginary.

It might also be pointed out that if the quadratic form $\sum_{r,s} d_{rs} \xi_r \xi_s$ is positive [2], $|\lambda| \leqq \sum_r d_{rr}$. The quadratic form, or the symmetric matrix D, is positive if and only if every principal minor of D is positive [1]. Also since D is real symmetric there exists a real number γ such that $\gamma I + D$ is positive [1], and the characteristic roots of $\gamma I + D$ are $\lambda + \gamma$.

4. Convergence. R. Bellman noted that the approximations can sometimes be sharpened by using the fact that the roots of A^2 are λ^2. Still more can be established for Theorems 0 and 2.

LEMMA 1. *If* A *is symmetric, has real positive elements, and the determinants of every two-rowed principal minor is zero, then the determinant of any minor of the second order is zero.*

Consider the minor

$$\begin{pmatrix} a_{rs} & a_{rv} \\ a_{us} & a_{uv} \end{pmatrix}.$$

By hypothesis,

$$a_{jj}a_{kk} = a_{jk}a_{kj} = a_{jk}^2 = a_{kj}^2.$$

Therefore,

$$\begin{aligned} a_{rs}a_{uv} &= (a_{rr}a_{ss})^{1/2}(a_{uu}a_{vv})^{1/2} \\ &= (a_{rr}a_{vv})^{1/2}(a_{uu}a_{ss})^{1/2} \\ &= a_{rv}a_{us}. \end{aligned}$$

LEMMA 2. *If A satisfies the conditions of Lemma* 1, *then the characteristic roots are* $\lambda_1 = \sum_r a_{rr}, \lambda_2 = \cdots = \lambda_n = 0$.

Since the determinants of all two-rowed minors are zero, the determinants of all minors of higher order are zero. Therefore, the characteristic equation reduces to

$$\lambda^n - \left(\sum_r a_{rr} \right) \lambda^{n-1} = 0,$$

as desired.

LEMMA 3. *Applying Theorem* 0 *to* A *exp* $\{2^{q-1}\}$, $q = 1, 2, \cdots$, *yields a monotone convergent sequence of numbers* $(\sum_{r,s} |a_{rs}^{(q)}|^2)^{1/2}$ *where* $a_{rs}^{(q)}$ *is an element of the matrix* A *exp* $\{2^{q-1}\}$. *If A satisfies the conditions of Lemma* 1, *the upper limit obtained for* $q+1$ *is the same as the upper limit obtained for* q.

The characteristic roots of A^2 are λ_r^2. Also,

$$A^2 = \left(\sum_k a_{rk}a_{ks} \right).$$

Hence, using Theorem 0, and Schwarz's Inequality,

$$\begin{aligned} |\lambda|^4 &\leqq \sum_{r,s} \left| \sum_k a_{rk}a_{ks} \right|^2 \\ &\leqq \sum_{r,s} \sum_k |a_{rk}|^2 \cdot \sum_l |a_{ls}|^2 \\ &= \sum_r \sum_k |a_{rk}|^2 \cdot \sum_l \sum_s |a_{ls}|^2 \\ &= \left(\sum_{r,s} |a_{rs}|^2 \right)^2. \end{aligned}$$

Now, if A satisfies the conditions of Lemma 1, we have,

$$
\begin{aligned}
|\lambda| &\leq \Big(\sum_{r,s} a_{rs}^2 \Big)^{1/2} \\
&= (a_{11}^2 + a_{22}^2 + \cdots + a_{nn}^2 + 2a_{12}^2 + \cdots + 2a_{n-1,n}^2)^{1/2} \\
&= (a_{11}^2 + a_{22}^2 + \cdots + a_{nn}^2 + 2a_{11}a_{22} + \cdots + 2a_{n-1,n}a_{nn})^{1/2} \\
&= a_{11} + a_{22} + \cdots + a_{nn},
\end{aligned}
$$

which, by Lemma 2, is the exact value of the non-zero characteristic root. Hence in this case A^2 gives the same approximation as A.

The process can be reapplied to A^2 and A^4, *etc.*

LEMMA 4. *If* $PA\overline{P}' = B$, $P\overline{P}' = I$, *then*

$$\sum_{r,s} |a_{rs}|^2 = \sum_{r,s} |b_{rs}|^2.$$

Consider first the matrix equation

$$QC = E,$$

where $Q = (y_{rs} + iz_{rs})$, $C = (c_{rs} + id_{rs})$, and $E = (e_{rs})$, and $Q\overline{Q}' = I$, with the y_{rs}, z_{rs}, c_{rs}, and d_{rs} real numbers.

By a direct multiplication, and use of the fact that $\overline{Q}'Q = I$, it is fairly easily seen that

$$\sum_{r} |e_{rs}|^2 = \sum_{r} (c_{rs}^2 + d_{rs}^2),$$

and hence that

$$\sum_{r,s} |e_{rs}|^2 = \sum_{r,s} (c_{rs}^2 + d_{rs}^2).$$

Multiplying the matrix E on the right by $\overline{Q}'$ and proceeding in a similar manner, the lemma follows.

THEOREM 3. *The convergent sequence of Lemma 3 converges to the value of the dominant characteristic root.*

For there exists a unitary matrix P which transforms A into a triangular matrix T, and moreover displays the characteristic roots in any desired order along the principal diagonal. (We may therefore assume them to be arranged in order of descending magnitude along the principal diagonal.) Coupling this with the result of Lemma 4, it is seen that any result concerning the characteristic roots of T and $\sum_{r,s} |t_{rs}|^2$ will be equally valid for A.

Suppose $|t_{11}| = max(|\lambda_r|)$. It is sufficient to indicate the proof for a second order matrix. Then

$$T = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}.$$

Consider also

$$T'' = \begin{pmatrix} |t_{11}| & |t_{12}| \\ 0 & |t_{11}| \end{pmatrix}.$$

If the sequence of numbers obtained by applying Theorem 0 to T'' converges to $|t_{11}|$, then it will certainly converge to $|t_{11}|$ when applied to T, since $|t_{22}| \leqq |t_{11}|$.

Now if $p = 2^q$,

$$(T'')^p = \begin{pmatrix} |t_{11}|^p & p|t_{11}|^{p-1} \cdot |t_{12}| \\ 0 & |t_{11}|^p \end{pmatrix}, \qquad q = 1, 2, \cdots.$$

Applying Theorem 0, we have

$$|\lambda| \leqq (|t_{11}|^{2p} + p^2|t_{11}|^{2p-2}|t_{12}|^2 + |t_{11}|^{2p})^{1/2p}. \tag{3}$$

The right-hand member of (3) approaches $|t_{11}|$ as p approaches infinity, as desired.

The only difficulty in considering matrices T of higher order is the complexity of notation encountered in forming T^p.

THEOREM 4. *Applying Theorem 2 to the sequence of matrices $A^{\exp\{2^{q-1}\}}$, $q = 1, 2, \cdots$, yields a sequence of numbers which converge to the dominant characteristic root.*

For evidently Theorem 2 applied to A yields a number not greater than that obtained using Theorem 0.

However the sequence is not necessarily monotone. Consider

$$A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}.$$

Theorem 2 applied to A and A^2 gives $|\lambda| \leqq 3$ and $|\lambda| \leqq (99)^{1/4}$, respectively.

5. Applications. As was pointed out in the previous paper [6], any number giving an upper limit for the dominant characteristic root will, when raised to the nth power, give an upper limit for the absolute value of the determinant.

In this connection we also state

THEOREM 5. *Let B_r denote the maximum absolute value of elements in the rth row, C_s for the sth column. Then*

$$|\det(A)| \leq n! \prod_r B_r.$$

A similar inequality holds for the C_s.

This follows from the fact that there are $n!$ elements of the determinant of A of the form $a_{j1}a_{k2} \cdots a_{ln}$, and

$$|a_{j1}a_{k2} \cdots a_{ln}| \leqq B_1B_2 \cdots B_n.$$

Since the characteristic equation of the matrix

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_2 & -a_1 \end{pmatrix}$$

is $$\lambda^n + a_1\lambda^{n-1} + \cdots + a_n = 0,$$

Theorem 4 can be used in calculating the dominant root of any polymonial equation.

Also Theorem 4 seems to be preferable to the method derived from the use of Sylvester's Theorem [7] in that all matrices are treated in exactly the same manner regardless of the types of characteristic roots involved.

Bibliography

1. Albert, Modern Higher Algebra, Univ. of Chicago Press, 1936, pp. 116–117.
2. Bateman, Partial Differential Equations, Dover Press, 1944, pp. 37–39.
3. Bendixson, Sur les racines d'une équation fondamentale, Acta Math., vol. 25, 1902, pp. 359–365.
4. Bromwich, On the roots of the characteristic equation of a linear substitution, Acta Math., vol. 30, 1906, pp. 295–304.
5. Browne, The characteristic roots of a matrix. Bull. Amer. Math. Soc., vol. 36, 1930, pp. 705–710.
6. Farnell, Limits for the characteristic roots of a matrix, Bull. Amer. Math. Soc., vol. 50, 1944, pp. 789–794.
7. Frazer, Duncan, and Collar, Elementary Matrices, Cambridge, 1938, pp. 134–138.
8. Hausdorff, Der Wertvorrat einer Bilinearform, Math. Zeit., vol. 3, 1919, pp. 314–316.
9. Hirsch, Sur les racines d'une équation fondamentale, Acta Math., vol. 25, 1902, pp. 367–370.
10. Parker, The characteristic roots of a matrix, Duke Math. J., vol. 3, 1937, pp. 484–487.
11. Toeplitz, Das algebraische Analogon zu einem Satze von Fejér, Math. Zeit., vol. 2, 1918, pp. 187–197.

Arithmetic in Literature. He brought up all his heaviest artillery of logic and mathematics, drawing forth his pencil and notebook to figure just how much a given sum of money could be increased if it was shrewdly invested now in this or in that piece of property, and then sold when the time was right.—Thomas Wolfe, *You Can't Go Home Again.*

And having burnished his arms, he sat down patiently to compute how much a half a dollar per diem would amount to at the end of a six-months campaign; and when he had settled that problem, proceeded to the more abtruse calculations necessary for drawing up a brigade of two thousand men on the principle of extracting the square root.—Scott, *A Legend of Montrose.*

It was computed by an experienced arithmetician, that there was as much two penny ale consumed on the discussion as would have floated a first-rate man-of-war.—Scott, *Heart of Mid-Lothian.*—*E. D. Schell.*

A CIRCLE COVERING THEOREM

A. W. GOODMAN, Republic Aviation Corporation
R. E. GOODMAN, Queens College

1. Introduction. The following theorem was conjectured by Erdös:

THEOREM 1. *Let the circles $C_1, \cdots, C_n$, with radii $r_1, \cdots, r_n$, lie in a plane and have the following property: No line of the plane divides the circles into two non-empty sets without touching or intersecting at least one circle. Then the circles $C_1, \cdots, C_n$ can be covered by a circle of radius $r = \sum_{j=1}^{n} r_j$.*

The theorem is trivial when $n = 2$. To the best of our knowledge, however, no proof has been given for any $n > 2$. We shall give here an analytic proof of Theorem 1 which can easily be generalized to give the analogous theorem for euclidean space of k dimensions. This proof depends on the following lemma.

2. Lemma. *Let $I_1, \cdots, I_n$ be intervals* on a line L, of lengths $2r_1, \cdots, 2r_n$ and with midpoints $P_1, \cdots, P_n$, and such that $\sum_{j=1}^{n} I_j = I$ is a single interval AB. Let $\phi(P) = \sum_{j=1}^{n} \overline{PP_j}^2 r_j$, where P is any point of L. Then $\phi(P)$ has an absolute minimum value for some point $P = P_0$. Furthermore, I is covered by the interval J with P_0 as center and length $2\sum_{j=1}^{n} r_j$.*

Proof. Let A be regarded as the left end of the interval AB. Let $I_1, \cdots, I_n$ be numbered in the order in which the left ends of these intervals occur on L, proceeding from A to B. Let s_j be the distance AP_j, and let s be the distance AP. Then

$$PP_j = \pm (s - s_j),$$

so that

$$\phi(P) = \sum_{j=1}^{n} \overline{PP_j}^2 r_j = \sum_{j=1}^{n} (s - s_j)^2 r_j = g(s).$$

Thus

$$g'(s) = \sum_{j=1}^{n} 2(s - s_j) r_j = 0$$

when

$$s = s_0 = \sum_{j=1}^{n} s_j r_j \Big/ \sum_{j=1}^{n} r_j,$$

and $g(s_0)$ is the minimum of $g(s)$. Now

$$s_j \leqq 2r_1 + 2r_2 + \cdots + 2r_{j-1} + r_j,$$

so that

* In this paper all sets are considered as closed.

$$s_0 \leqq \sum_{j=1}^{n} r_j(2r_1 + \cdots + 2r_{j-1} + r_j) \Big/ \sum_{j=1}^{n} r_j$$

$$= \Big(\sum_{j=1}^{n} r_j\Big)^2 \Big/ \sum_{j=1}^{n} r_j = \sum_{j=1}^{n} r_j.$$

If now P_0 is the point such that $AP_0 = s_0$, then the above inequality implies that A is covered by the interval J.

By renumbering the intervals $I_1, \cdots, I_n$ in the order in which their right ends occur, proceeding from B to A, we prove similarly that B is covered by J.

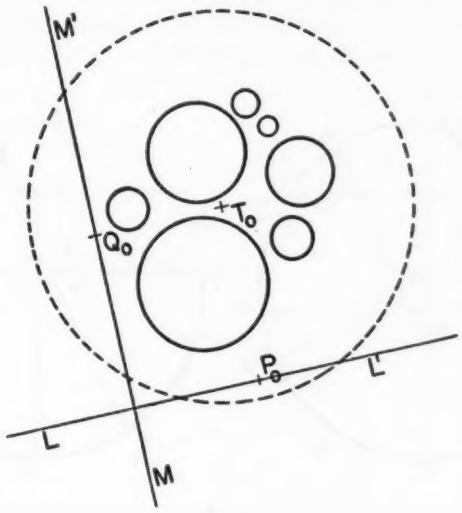


FIG. 1.

3. Proof of Theorem 1. Let $O_1, \cdots, O_n$ be the centers of $C_1, \cdots, C_n$. Define the function

$$f(T) = \sum_{j=1}^{n} \overline{TO}_j^2,$$

where T is any point in the plane. Let LL' and MM' be any two mutually perpendicular lines in the plane. (See Fig. 1.) Let P_j and Q_j be the projections of O_j on LL' and MM', respectively, and P and Q the projections of T on LL' and MM'. Then

$$\overline{TO}_j^2 = \overline{PP}_j^2 + \overline{QQ}_j^2,$$

so that

$$f(T) = \phi(P) + \psi(Q),$$

where

$$\phi(P) = \sum_{j=1}^{n} \overline{PP'_j}^2,$$

$$\psi(Q) = \sum_{j=1}^{n} \overline{QQ'_j}^2.$$

Since the circles $C_1, \cdots, C_n$ are nonseparable by any line in the plane, their projections on LL' and MM' are intervals satisfying the conditions of the lemma. Let P_0 on LL' and Q_0 on MM' be the points such that $\phi(P_0)$ and $\psi(Q_0)$ are the minimum values of $\phi(P)$ and $\psi(Q)$. Let T_0 be the point of intersection of the line through P_0 and parallel to MM' and the line through Q_0 and parallel to LL'. Let C be the circle with center at T_0 and radius r. By the lemma, the projection of C covers the projections of $C_1, \cdots, C_n$ on LL' and MM'.

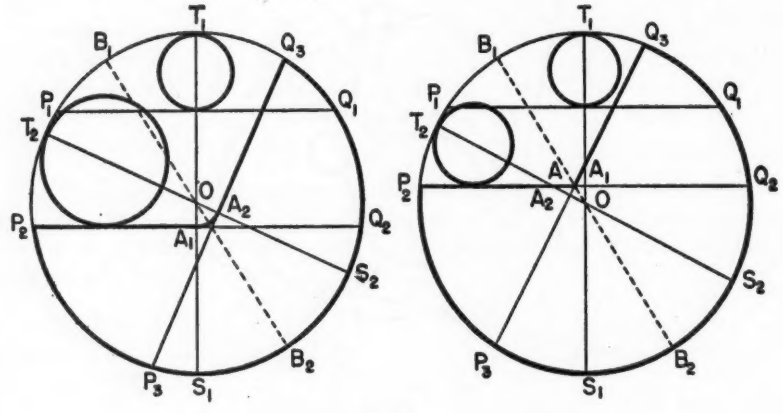


FIG. 2.

Furthermore, it is clear that the minimum of the function $f(T)$ is $f(T_0)$, and this minimizing property implies that the location of the point T_0 is independent of the particular pair of lines LL' and MM' used to construct T_0. Hence it is true that the projection of C on any line of the plane covers the projections of $C_1, \cdots, C_n$ on that line. Thus $C_1, \cdots, C_n$ are covered by every square circumscribed about C, whence $C_1, \cdots, C_n$ are covered by the product of these squares, that is, the circle C.

4. Geometric proof for a special case. For the special case $n=3$, we can give a purely geometric proof of Theorem 1, which seemingly cannot be generalized.

If C is the smallest circle which covers the given non-separable circles C_1, C_2, C_3, then either only two of the given circles are tangent to C or all three of the given circles are tangent to C. In the former case, the two points of tangency must lie at opposite ends of a diameter of C, and the theorem is trivial.

In the latter case, every semi-circumference of C contains one of the three points of tangency.

Let C_1, the smallest of the given circles, be tangent to C at T_1. (See Fig. 2.) Let chord P_1Q_1 be perpendicular to diameter T_1OS_1 of C, and tangent to C_1. Now P_1Q_1 cuts another circle, C_2. (All three given circles cannot lie in segment $P_1T_1Q_1$, since C_1 is the smallest of the given circles.) Let P_2Q_2 be the chord parallel to P_1Q_1 which is tangent to C_2 and does not intersect C_1, and let A_1 be the point of intersection of P_2Q_2 with T_1S_1. (Note, as in Fig. 2, that A_1 may be interior either to T_1O or to OS_1.) Let A_2 lie on T_2O with $T_1A_1=T_2A_2$. Let chord $P_3A_2Q_3$ be perpendicular to diameter T_2OS_2. Now T_3, the point of tangency of C_3 with C, must lie on that arc S_1S_2 which does not contain T_1 and T_2. If $2r_3 \geqq A_1S_1$, the theorem is true. If $2r_3 < A_1S_1$, then C_3 does not touch the heavy boundary $P_3A_1A_2Q_3$ (or P_3AQ_3). But this boundary is symmetric about the bisector B_1B_2 of $\sphericalangle T_1OT_2$, whence it is clear that C_3 can be separated from C_1 and C_2 by some line tangent to the arc A_1A_2 with O as center (or through A).

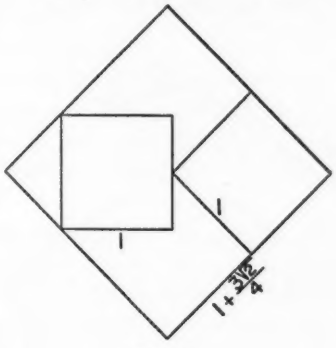


FIG. 3.

5. An analogous problem for convex bodies. Theorem 1 suggests consideration of the following problem:

Let $P_1, \cdots, P_n$ be given similar convex bodies with diameters $d_1, \cdots, d_n$. Let d be the diameter of the smallest convex body similar to the given convex bodies which can be placed so as to cover $P_1, \cdots, P_n$ whenever they are so situated in a plane that no line of the plane separates them into two non-empty sets without touching or intersecting at least one of them. Let $d=\lambda\sum_{j=1}^{n} d_j$. What is the value of λ?

Since $P_1, \cdots, P_n$ may always be placed with their diameters on a straight line and non-overlapping, it is clear that $\lambda \geqq 1$ in every case. Application of Theorem 1 shows at once that $\lambda \leqq \mu$ where μ is the ratio of the radius of the circumcircle to that of the incircle for the given set of convex bodies. It is clear that λ is arbitrarily large for sufficiently distorted figures. That λ may actually be >1, however, even for regular polygons is shown by the case of two congruent squares. From the arrangement in Fig. 3, $\lambda \geqq (4+3\sqrt{2})/8 > 1.03$.

If we specialize the above problem by requiring that $P_1, \cdots, P_n$ be not only similar but also homothetic, it seems reasonable to believe that $\lambda = 1$, although we have no proof of this conjecture.

The method of proof used for Theorem 1 immediately shows that $\lambda = 1$ for homothetic regular polygons for an even number of sides. Indeed, we may even relax the condition of non-separability of $P_1, \cdots, P_n$ to non-separability only by lines parallel to the sides of the given polygons, still obtaining the same covering polygon.

For homothetic regular polygons of an odd number of sides, however, the method of Theorem 1 does not produce results. And with the relaxed condition of non-separability for homothetic regular polygons of an odd number of sides, we find that $\lambda > 1$.

For let P be a regular polygon with vertices $V_1, \cdots, V_{2k+1}$. (See Fig. 4.) Let P_1 have vertices $U_1, \cdots, U_{2k+1}$, and let U_1U_2 lie on V_1V_2, with midpoints coinciding. Let the diameter of P_1 be large enough so that the line LL' through U_{k+3} and parallel to V_2V_3 and the line MM' through U_{k+2} and parallel to V_1V_2

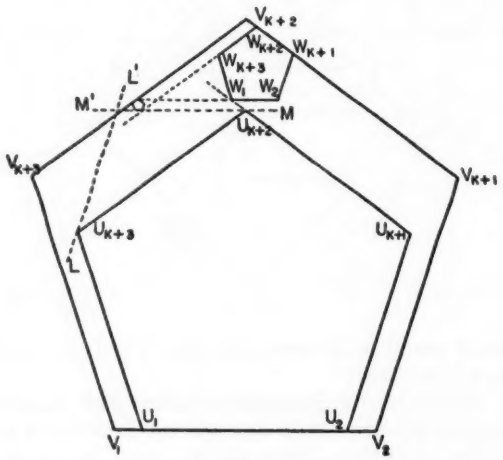


FIG. 4.

intersect outside P. Let P_2, with vertices $W_1, \cdots, W_{2k+1}$, have diameter d_2 equal to the distance between $V_{k+1}V_{k+2}$ and $U_{k+1}U_{k+2}$, and let P_2 be placed so that the distance between W_1W_2 and MM' is equal to the distance between $W_{k+2}W_{k+3}$ and $V_{k+2}V_{k+3}$, and call this distance d_3. Let P_3 have diameter d_3 and be placed with its base Y_1Y_2 on MM' and $Y_{k+2}Y_{k+3}$ on $V_{k+2}V_{k+3}$. It is clear, then, that P_1, P_2, P_3 cannot be separated by any line parallel to their sides and that P is the smallest polygon which will cover them. But $d > d_1 + d_2 + d_3$, so that $\lambda > 1$.

THE LAGRANGE IDENTITY AS A UNIFYING PRINCIPLE

LOUIS BRAND, University of Cincinnati

1. Introduction. Many important facts relating to linear differential equations of the second order can be systematized and unified by means of the *Lagrange identity* connecting adjoint differential expressions. If we write the reduced equation

$$P(u) = P_0(x)u'' + P_1(x)u' + P_2(x)u = 0, \tag{1}$$

the multipliers of $P(u)$ and the test for its exactness are at once apparent from this identity. When $P(u)$ is made self-adjoint by multiplication by a suitable factor $\lambda(x)$, the Lagrange identity for $\lambda P(u)$ leads at once to:

(a) The consideration of the Wronskian of two linearly independent solutions of the reduced equation;
(b) the Liouville differential equation satisfied by this Wronskian;
(c) the immediate deduction of a second solution of the reduced equation when a first particular solution is known;
(d) the general solution of the complete equation.

All these important elementary matters thus flow simply and naturally from the Lagrange identity, and without resort to artifices, such as "variation of parameters." This identity is also useful in dealing with equations of the Sturm-Liouville type. .

2. Adjoint differential expressions. We assume that the $P_0(x)$, $P_1(x)$, $P_2(x)$ are continuous, twice-differentiable functions of the real variable x in the interval $a \leqq x \leqq b$, and that $P_0 \neq 0$ in this interval. By transforming the term of $vP(u)$ by the rule for differentiating a product we obtain the *Lagrange identity*

$$vP(u) - uQ(v) = \frac{d}{dx} R(u, v), \tag{2}$$

where

$$R(u, v) = P_0(u'v - uv') + (P_1 - P_0')uv \tag{3}$$

is a bilinear expression of the first order in u, v, and

$$Q(v) = (P_0v)'' - (P_1v)' + P_2v \tag{4}$$

is known as the *adjoint* of $P(u)$.

From the *Lagrange identity* (2) it is readily shown that $Q(v)$ is the only differential expression which satisfies a relation of this form. Thus the identity establishes a unique association of differential expressions $P(u)$, $Q(v)$, which therefore are *adjoint to each other*.

If $P(u) = Q(u)$, $P(u)$ is said to be *self-adjoint*. In this case the Lagrange identity with $v = u$ requires that $R(u, u) = (P_1 - P_0')u^2$ be constant and hence

$$P_1 - P_0' = 0; \tag{5}$$

this necessary condition that $P(u)$ be self-adjoint is obviously sufficient.

If $P_1 - P_0' \neq 0$, a function $\lambda(x)$ which satisfies $(\lambda P_1) - (\lambda P_0)' = 0$ will make $\lambda P(u)$ self-adjoint; such a function is defined by

$$\lambda P_0 = \exp \int_c^x \frac{P_1}{P_0} dx \tag{6}$$

where c is a point of (a, b).

The Lagrange identity for the self-adjoint form $\lambda P(u)$ is

$$\lambda v P(u) - \lambda u P(v) = \frac{d}{dx} \{\lambda P_0 (u'v - uv')\}. \tag{7}$$

3. Multipliers. A function $v(x)$ is said to be a *multiplier* of $P(u)$ if $vP(u)$ is identically equal to the derivative of a linear differential expression of the first order in u. For any v, the right member of the Lagrange identity (3) is such a derivative, but the term $uQ(v)$ on the left can not be so expressed. We may therefore state

THEOREM 1. *A function v is a multiplier of $P(u)$ when and only when $Q(v) = 0$.*

We pass now from differential *expressions* to differential *equations* and call $P(u) = 0$, $Q(v) = 0$ *adjoint equations*. Then we may say that the multipliers of $P(u) = 0$ are precisely those functions which satisfy its adjoint $Q(v) = 0$.

If $P(u)$ is expressible as a derivative, the equation $P(u) = 0$ is said to be *exact*. In this case $P(u)$ admits the multiplier 1 and hence $Q(1) = 0$. The Lagrange identity, with $v = 1$, now gives

$$P(u) = \frac{d}{dx} \{P_0 u' + (P_1 - P_0')u\}. \tag{8}$$

In this case the *complete* linear equation

$$P(u) = f(x) \tag{9}$$

may be solved by quadratures: for its first integral

$$P_0 u' + (P_1 - P_0')u = \int f(x) dx + C,$$

is a linear equation of the first order.

If $P(u) = 0$ is not exact, $vP(u) = 0$ will be exact if v is a solution of $Q(v) = 0$. Thus if any non-zero solution of $Q(v) = 0$ is known, the general solution of (9) may be obtained by quadratures.

4. The Wronskian. If u_1 and u_2 are two solutions of $P(u) = 0$, let $v = u_1$, $u = u_2$ in (7); then

$$\frac{d}{dx} (\lambda P_0 W) = 0 \quad \text{where} \quad W = \begin{vmatrix} u_1 & u_2 \\ u_1' & u_2' \end{vmatrix}$$

is the *Wronskian* of the two solutions. On integration we obtain

$$\lambda P_0 W = (\lambda P_0 W)_{x=c} = W(c) \tag{10}$$

since $\lambda P_0 = 1$ when $x = c$; hence, from (6),

$$W = W(c) \exp\left(-\int_c^x \frac{P_1}{P_0}\,dx\right). \tag{11}$$

5. Solution of adjoint equation. If u_1 and u_2 are linearly independent solutions of $P(u) = 0$, their Wronskian $W \neq 0$. From (7), λu_1 and λu_2 are multipliers of $P(u) = 0$ and therefore linearly independent solutions of the adjoint equation $Q(v) = 0$. From (10) we have $\lambda = W(c)/P_0 W$; hence the

THEOREM 2. *If u_1, u_2 are linearly independent solutions of $P(u) = 0$ whose Wronskian is W,*

$$v_1 = u_1/P_0 W, \qquad v_2 = u_2/P_0 W \tag{12}$$

are linearly independent solutions of the adjoint equation $Q(v) = 0$.

A simple calculation shows that

$$W(u_1, u_2)W(v_1, v_2) = 1/P_0^2. \tag{13}$$

6. Solution of complete equation. If u_1 is a non-zero solution of $P(u) = 0$, we have from (7), with $v = u_1$,

$$\lambda u_1 P(u) = \frac{d}{dx}\{\lambda P_0(u'u_1 - uu_1')\}; \tag{14}$$

thus λu_1 is a multiplier of $P(u)$. The complete equation (9) may be written

$$\frac{d}{dx}\{\lambda P_0(u'u_1 - uu_1')\} = \lambda u_1 f(x); \tag{15}$$

hence the fundamental

THEOREM 3. *The general solution of the complete equation can be found by quadratures if any non-zero solution of the reduced equation is known.*

A first integration of (15) gives

$$\lambda P_0(u'u_1 - uu_1') = \int_c^x \lambda(t)u_1(t)f(t)\,dt + B. \tag{16}$$

On dividing this by $\lambda P_0 u_1^2$ we have

$$\frac{d}{dx}\left(\frac{u}{u_1}\right) = \frac{1}{\lambda P_0 u_1^2}\int_c^x \lambda u_1 f\,dt + \frac{B}{\lambda P_0 u_1^2}.$$

A second integration yields the complete solution

$$u = u_1 \int_c^x \frac{ds}{\lambda P_0 u_1^2} \int_c^s \lambda u_1 f \, dt + B u_1 \int_c^x \frac{dt}{\lambda P_0 u_1^2} + A u_1. \tag{17}$$

This discloses that the reduced equation has the second particular solution

$$u_2(x) = u_1(x) \int_c^x \frac{dt}{\lambda P_0 u_1^2}. \tag{18}$$

Since $u_2'(c) = 1/u_1(c)$, $W(u_1, u_2) = 1$ when $x = c$.

The iterated integral in (17) may be written

$$U(x) = u_1 \int_c^x \left(\frac{u_2}{u_1}\right)' ds \int_c^s \lambda u_1 f \, dt. \tag{19}$$

On integration by parts it assumes the form

$$\begin{aligned} U(x) &= u_2(x) \int_c^x \lambda u_1 f \, dt - u_1(x) \int_c^x \lambda u_2 f \, dt \\ &= \int_c^x \lambda(t) f(t) \begin{vmatrix} u_1(t) & u_2(t) \\ u_1(x) & u_2(x) \end{vmatrix} dt \end{aligned} \tag{20}$$

where $\lambda(t) = 1/P_0(t)W(t)$ from (10). $U(x)$ is evidently a particular solution of the complete equation; it is determined by the conditions $U(c) = 0$, $U'(c) = 0$. The solution of the complete equation,

$$u = A u_1(x) + B u_2(x) + U(x) \tag{21}$$

agrees with that obtained by Lagrange's method of "variation of parameters."

If we have two independent solutions u_1, u_2 of the reduced equation, we can deduce the solution $U(x)$ by obtaining two first integrals of (9) of the form (16) and eliminating u'. Putting $\lambda = W(c)/P_0 W$ in (16), we have

$$\frac{1}{W}(u_i u' - u_i' u) = \int_c^x \frac{u_i(t)}{P_0(t)W(t)} f(t) dt + C_i \qquad (i = 1, 2).$$

Elimination of u' from these equations again gives (21).

MAGIC RECTANGLES MODULO p

E. J. FINAN, Catholic University

1. Introduction. Since remote times magic configurations of integers have been of interest to mathematicians. There are magic cubes, rectangles, spheres, pencils, crosses, squares, *etc.* The best known and most interesting of these is the magic square which still merits attention. Its most obvious property of interest—its magic property—is that the sums of the elements in each row, each column, and each principal diagonal are the same.

The purpose of this paper is to discuss a rectangular array of positive integers which is associated with elementary number theory. One of the properties of the rectangle, although not the most useful, is that the sum of the integers in any row or column is zero modulo p. This justifies the term Magic Rectangle. Some of the properties which are listed below as theorems are restatements, special cases, or generalizations of well known theorems. Others are peculiar to this particular configuration.

The results of section 4 may be of assistance in performing some of the calculations one meets in elementary number theory. This is true in particular when one modulus is used at length since the rectangles $R_{t,k}$ and $A_{t,k}$ yield with very little work the primitive roots, the nth power residues, the reciprocals, and the solution of $ax \equiv b$ all modulo p. However if only one of the above quantities, for example a primitive root, is wanted, it is not advisable to construct $R_{t,k}$.

Unless otherwise stated all congruences are modulo p.

2. Law of formation. Let p be an odd prime for which $\phi(p) = p-1 = kt$, where $(k, t) = 1$ and k is even. Let α and β designate the smallest* positive integers that belong to k and t respectively mod p. The rectangular array whose element in the (i, j) cell or position is the least positive residue of $\beta^i\alpha^j$ mod $p (i = 1, 2, \cdots, t;$ $j = 1, 2, \cdots, k)$, will be called the magic rectangle $R_{t,k}$. It is uniquely determined by t and k for a given p.

For example, let $p = 43$, $t = 7$ and $k = 6$. Then $\alpha = 7$, $\beta = 4$, and $R_{7,6}$ is the first of the two rectangles appearing below.

28	24	39	15	19	4
26	10	27	17	33	16
18	40	22	25	3	21
29	31	2	14	12	41
30	38	8	13	5	35
34	23	32	9	20	11
7	6	42	36	37	1

1	7	6	42	36	37	1
4	28	24	39	15	19	4
16	26	10	27	17	33	16
21	18	40	22	25	3	21
41	29	31	2	14	12	41
35	30	38	8	13	5	35
11	34	23	32	9	20	11
1	7	6	42	36	37	1

3. Properties. Some of the interesting properties of such a rectangular array are included in the following theorems.

THEOREM 1. *Each of the numbers* $1, 2, \cdots, kt = p-1$ *appears in* $R_{t,k}$ *exactly once.*

* The restriction that k be even and that α and β be smallest are imposed for convenience in order to make $R_{t,k}$ unique and to make some of the proofs easier to word.

Since each of the kt cells is filled it is sufficient to show the entries are distinct. Assume $\beta^i\alpha^j \equiv \beta^r\alpha^s$ where $j \leqq s$. Then $\beta^i \equiv \beta^r\alpha^{s-j}$. After raising both sides of the last congruence to the tth power it is evident k divides $s-j$ because $(k, t)=1$ and α belongs to k. Hence $j=s$ and $\beta^i \equiv \beta^r$. But this is impossible unless $i=r$.

THEOREM 2. *All the elementary symmetric functions of the numbers in each row or column of $R_{t,k}$ excepting the product is congruent to zero. The product of the elements in the ith row is congruent to $-\beta^{ik}$ and the product of those in the jth column is congruent to α^{jt}.*

Since β belongs to t, the elements in the last row are $\alpha, \alpha^2, \cdots, \alpha^k=1$. They are roots of $x^k \equiv 1$. There are no other roots because a congruence with prime modulus cannot have more roots than its degree. The elements in the ith row are the products of the corresponding elements in the last row by β^i. Hence they are all of the roots of the congruence $x^k \equiv \beta^{ik}$. In the same way the elements of the jth column are all of the roots of the congruence $x^t \equiv \alpha^{jt}$. Theorem 2 follows from the above observations and the relations between the coefficients of an equation and the elementary symmetric functions of its roots.

COROLLARY. *If the product of the elements in the ith row of $R_{t,k}$ is $\theta_i (i=1, 2, \cdots, t)$ then the roots of $x^t \equiv -1$ are θ_i.*

These products are $-\beta^k, -\beta^{2k}, \cdots, -\beta^{tk}$, which satisfy $x^t \equiv -1$ because β belongs to t and t is odd.

When $k=p-1$ and $t=1$ the rectangle consists of one row. The corollary is then a statement of Wilson's Theorem. Hence Theorem 2 is a generalization of Wilson's Theorem unless $p=2^{2n}+1$. In this case it yields Wilson's Theorem.

COROLLARY. *If the product of the elements in the jth column of $R_{t,k}$ is $\pi_j (j=1, 2, \cdots, k)$ then the roots of $x^k \equiv 1$ are π_j.*

THEOREM 3. *Each row (column) of $R_{t,k}$ is a multiple of every row (column).*

For example, the product of the ith row by β^{k-i} is the kth row.

For the next theorems it is convenient to denote the rows and columns of $R_{t,k}$ as $r_i (i=1, 2, \cdots, t)$ and $c_j (j=1, 2, \cdots, k)$ respectively. Also define c_0 as an additional column consisting of t zeros. The sum c_r+c_s will be defined as follows. Select any two elements, one from c_r and one from c_s, lying in the same row. Let their sum modulo p be in c_m. Then $c_r+c_s=c_m$. It remains to show the sum is well defined, *i.e.*, in this instance, that it is unique. In equation form this amounts to showing that if $\beta^i\alpha^r+\beta^i\alpha^s=\beta^i\alpha^m$ then $\beta^k\alpha^r+\beta^k\alpha^s=\beta^n\alpha^m$ for some n. Multiply the first equation by β^{k-i}. This gives $\beta^k\alpha^r+\beta^k\alpha^s=\beta^{i+k-i}\alpha^m$ which is in c_m. Hence the c_m is determined uniquely. Since the elements in the last row of $R_{t,k}$ are the roots of $x^k \equiv 1$ where k is even, these roots may be paired so that the sum of each pair is zero modulo p. Consequently for any given column c_s there exists a column c_b so that $c_s+c_b=c_0$. Hence the $c_j (j=0, 1, \cdots, k)$ form a set of $k+1$ elements which is closed under addition, contains the identity,

contains the inverse of each element, but does not obey the associative law. For example, in the above illustration $(c_1+c_2)+c_6=c_4+c_6=c_5$ while $c_1+(c_2+c_6)=c_1+c_1=c_4$.

Similarly the product c_rc_s of two columns may be defined as follows. Select two elements $\beta^i\alpha^r$ and $\beta^i\alpha^s$ lying in the same row of $R_{t,k}$. Let their product modulo p be in c_m. Then $c_rc_s=c_m$. Again one must show this product is unique. This amounts to proving that if $(\beta^i\alpha^r)(\beta^i\alpha^s)=\beta^t\alpha^m$ then $(\beta^q\alpha^r)(\beta^q\alpha^s)_k\beta^n\alpha^m$ for some n. However this is obvious since the exponent on α in both equations is $r+s$. Since $\alpha^k\equiv 1 \bmod p$ the exponents on α are reduced modulo k. Also since the product of two columns may be obtained by the addition modulo k of the exponents on α the $c_j(j=1, 2, \cdots, k)$ form a group under multiplication which is isomorphic with the cyclic additive group whose elements are $1, 2, \cdots, k$ with addition performed modulo k. Call this group C. In the same way it can be shown that the $r_i(i=1, 2, \cdots, t)$ form a group R under multiplication which is isomorphic with the cyclic additive group of order t.

One may form the symbols (r_i, c_j), $i=1, 2, \cdots t$; $j=1, 2, \cdots k$, and define the product of two such by the equation $(r_q, c_d)(r_m, c_s)=(r_{q+m}, c_{d+s})$ where the addition of the subscripts on r and c is performed modulo t and k respectively. These $kt=p-1$ symbols (r_i, c_j) form a group P which is the direct product* $R\times C$ of R and C. Since it may be generated by (r_1, c_1) it is cyclic. Also the $p-1$ elements of $R_{t,k}$ form a group G under multiplication modulo p. It may be generated by $\beta\alpha$. Hence P and G are isomorphic. One isomorphic correspondence between the two is $\beta^i\alpha^j\leftrightarrow(r_i, c_j)$. These results are stated in the next two theorems.

THEOREM 4. *The set $c_j(j=0, 1, \cdots, k)$ is closed under addition, contains the identity, contains the unique inverse of each element but does not obey the associative law.*

THEOREM 5. *The $r_i(i=1, 2, \cdots, t)$ and $c_j(j=1, 2, \cdots, k)$ form cyclic groups under multiplication. Their direct product is isomorphic with the multiplicative group modulo p composed of the elements of $R_{t,k}$.*

Let $A_{t,k}$ be a rectangular array of $t+1$ rows and $k+1$ columns formed from $R_{t,k}$ as follows. First place an additional row consisting of $\alpha, \alpha^2, \cdots, \alpha^k=1$ at the top of $R_{t,k}$. Then place an additional column consisting of $\beta, \beta^2, \cdots, \beta^t=1$ at the left of $R_{t,k}$. Finally place unity in the (1,1) position of $A_{t,k}$. The second rectangle displayed above is $A_{7,6}$. It is convenient to call the elements in the added row and column $\beta^0\alpha^j(j=0, 1, \cdots, k)$ and $\beta^i\alpha^0(i=0, 1, \cdots, t)$ respectively. Let O be the geometric center of the rectangle determined by $A_{t,k}$. If the reflection of $\beta^r\alpha^s$ in O is $\beta^b\alpha^a$ then $b+r=t$ and $a+s=k$. This proves

THEOREM 6. *The inverse modulo p of any number is its reflection in the center of $A_{t,k}$.*

* See, for example, MacDuffee, Introduction to Abstract Algebra, p. 67.

4. Applications. Once $A_{t,k}$ has been constructed for a given p, a good many of the elementary number theory results are easily obtained. This refers to both the theorems and calculations. A few illustrations will be given.

If a is in cell (i, j) and b is in cell (r, s) then ab is in cell $(i+r, j+s)$ where additions are performed modulo t and k respectively.

The inverse of a is obtained by inspection from $A_{t,k}$ as explained in Theorem 6.

The solution of $ax \equiv b \bmod p$ is $x = a^{-1}b$ which may be obtained from $A_{t,k}$.

THEOREM 7. *A necessary and sufficient condition that $x^n \equiv \beta^i\alpha^j$ be solvable is that $d_1 = (t, n)$ divide i and $d_2 = (k, n)$ divide j. There are then d_1d_2 solutions.*

This follows from the fact that the nth root of β^i exists if and only if there is an x such that $tx + i \equiv 0 \bmod n$. This x exists if and only if $d_1 = (t, n)$ divides i and there are then d_1 values of x. In the same way there are $d_2 = (k, n)$ nth roots of α^j if d_2 divides j. This yields a total of d_1d_2 solutions.

An equivalent result, Euler's Criterion,* may be stated as follows: "If $d = (n, p-1)$, a necessary and sufficient condition that $x^n \equiv b$ be solvable is that $b^{(p-1)/d} \equiv 1$. There are then exactly d roots." To show they are equivalent first assume d_1 divides i and d_2 divides j. Then $t = t_1d_1$, $k = k_2d_2$, $n = n_1d_1 = n_2d_2$, $i = i_1d_1$, and $j = j_2d_2$ where $(d_1, d_2) = 1$ since $(k, t) = 1$. The congruence $b^{(p-1)/d} \equiv 1$ may then be written $(\beta^i\alpha^j)^{t_1k_2} \equiv 1$ or $\beta^{t_1k_2i_1d_1}\alpha^{t_1k_2j_2d_2} \equiv 1$ which is evident since the exponents on β and α are multiples of t and k respectively. Hence if d_1 divides i and d_2 divides j then $b^{(p-1)/d} \equiv 1$.

Now assume $b^{(p-1)/d} = (\beta^i\alpha^j)^{(p-1)/d} \equiv 1$ where $d = (n, kt)$ and $kt = p-1$. Since $(k, t) = 1$ one can write $d = d_1d_2$ where $d_1 = (t, n)$, $d_2 = (k, n)$, and $(d_1, d_2) = 1$. Then using same notation as above, from $(\beta^i\alpha^j)^{(p-1)/d} \equiv 1$ follows $\beta^{t_1k_2i} \equiv \alpha^{t_1k_2j} \equiv 1$. Hence there exists an integer s so that $t_1k_2i = st = st_1d_1$ from which d_1 divides i since $(d_1, k_2) = 1$. In the same it follows that d_2 divides j.

Theorem 7 enables one to obtain the nth power residues.† For example, to obtain the cubic residues modulo 43 let $n = 3$, $i = 7$, and $k = 6$. Since $(7, 3) = 1$ the residues may be in any row. Since $(6, 3) = 3$ then j must be a multiple of 3. Hence the cubic residues are the 14 numbers in the 3rd and last columns of $R_{7,6}$. Of course, the quadratic residues are those numbers in the 2nd, 4th and 6th columns.

The primitive roots‡ modulo p are those numbers $\beta^i\alpha^j$ for which $(i, t) = (j, k) = 1$. For $p = 43$ they are in the first and fifth columns and in all rows except the last of $R_{7,6}$.

By replacing p and $p-1$ by m and $\phi(m)$ respectively it is possible to construct $R_{t,k}$ using only those integers in a reduced residue system. The procedure parallels the above.

---

* *Cf.* L. W. Reid, The Elements of the Theory of Algebraic Numbers, p. 115.

† *Cf.* Uspensky and Heaslet, Elementary Number Theory, p. 202 et seq.

‡ *Ibid.*, p. 232. The above method does not shorten the labor necessary to find a primitive root.

DISCUSSIONS AND NOTES

EDITED BY MARIE J. WEISS, Sophie Newcomb College, New Orleans 18, La.

The Department of Discussions and Notes is open to all forms of activity in college mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

RATIONAL VALUES OF TRIGONOMETRIC FUNCTIONS

J. M. H. OLMSTED, University of Minnesota

A recent article* pointed out that $\cos x$ is an algebraic number whenever x is equal to an integral number of seconds. An immediate extension of this is that all six standard trigonometric functions have algebraic values for angles measured rationally in degrees. An earlier article† gave a more advanced discussion of these algebraic numbers. It is the purpose of this note to present an elementary proof of the fact, apparently not widely known, that these numbers are scarcely ever rational—in fact only for the very familiar values associated with the sequence $0°, 30°, 45°, \cdots$.

THEOREM. *If x is rational in degrees, then the only possible rational values of the trigonometric functions are: $\sin x$, $\cos x = 0, \pm 1/2, \pm 1$; $\sec x$, $\csc x = \pm 1, \pm 2$; $\tan x$, $\cot x = 0, \pm 1$.*

The proof can obviously be limited to the cosine and tangent. Assume that $\cos x$ is rational but not equal to one of the given values, and that kx equals a multiple of $360°$, where k is an integer. Then for any integer n, knx is a multiple of $360°$. Now $\cos knx$ can be expressed as a polynomial in $\cos nx$ with integral coefficients and with leading coefficient equal to 2^{k-1}. Since $\cos nx$ is rational and satisfies the equation obtained by equating this polynomial to 1, therefore when $\cos nx$ is expressed as a rational fraction in lowest terms, the denominator must be a factor of 2^{k-1}, and incidentally a power of 2. A contradiction is obtained by showing that this denominator of $\cos nx$ can be made arbitrarily large. Accordingly, let $\cos \alpha = p/q$, where p is odd and q is a power of 2 greater than the first. Then $\cos 2\alpha = p'/q'$, where $p' = p^2 - q^2/2$, an odd number, and $q' = q^2/2$, a power of $2 > q$. Thus the terms of the sequence obtained by repeated doubling of the angle, $\cos x$, $\cos 2x$, $\cos 4x$, $\cdots$, when expressed as rational fractions in lowest terms, have successively larger denominators.

Proof for the tangent follows essentially the same pattern. In this case $\tan knx$ can be expressed as the quotient of two polynomials in $\tan nx$, each having integral coefficients, and the numerator having leading coefficient 1 or k. Since $\tan nx$ is rational (if finite) and satisfies the equation obtained by equating the numerator polynomial to 0, therefore when $\tan nx$ is expressed as a rational

* R. W. Hamming, The transcendental character of cos x, this MONTHLY, vol. 52, 1945, pp. 336–337.

† D. H. Lehmer, A note on trigonometric algebraic numbers, this MONTHLY, vol. 40, 1933, pp. 165–166.

fraction in lowest terms, the denominator must be a factor of k. Let $\tan \alpha = p/q$, where $p \neq \pm q$, $p \neq 0$, $q \neq 0$, and the fraction is in lowest terms. Then $\tan 2\alpha = p'/q'$, where $p' = 2pq \neq 0$ and $q' = q^2 - p^2 \neq 0$. It is impossible for p' to equal $\pm q'$ since the largest possible common factor of p' and q' is 2, this occurring only when p and q are both odd. In any case, when p'/q' is expressed in lowest terms, the resulting fraction is of the same type as p/q, and the new denominator is numerically greater than the first. Again a contradiction is provided by a sequence obtained by repeated doubling.

THE AREA OF A TRIANGLE AS A FUNCTION OF THE SIDES*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. Historical remarks. The first mention of the rule giving the area of a triangle as a function of the three sides is found in the works of Heron of Alexandria (1st century). Although it is now believed that this rule pre-dates Heron, demonstrations of it are in his two works, *Metrics* and *Treatise on the Diopter*.

In the book of the three Arabian brothers, Mohammed, Ahmed, and Alhasan, (9th century) we encounter a new demonstration, the first which came to us in Europe. It was reproduced by Leonardo of Pisa in his *Practical Geometry* (1220) and then by Jordanus Nemorarius (13th century), and by most of the geometers of the Renaissance. It is curious that Heron, the Hindus, as well as all the authors we have cited, made an application of this rule to the triangle of sides 13, 14, and 15, whose area is 84. One is led to ask if these three numbers have a common origin, but, as Chasles had observed, the Greeks, the Hindus, and the Arabs may very well have separately become aware of the fact that 13, 14, 15 are the smallest integers which give a rational area for an acute angled triangle.

One finds, still later, other new proofs of the rule by Newton in his *Universal Arithmetic* (1707); by Euler in the *Recent Commentaries of Petersburg* (v. I, 1747, p. 48); by Boscovich in volume V of his *Works* concerning optics and astronomy (1785). This last demonstration is obtained by trigonometric considerations.

2. New demonstration. The author has previously given a very short geometrical demonstration of the formula under consideration (*Mathesis*, 1931, p. 27), and here is another equally simple.

Being given a triangle ABC, ($BC = a$, $CA = b$, $AB = c$, $a+b+c = 2p$), let (see figure) B' and B'', C' and C'' be the orthogonal projections of the vertices B and C on the bisectors AD and AD' of angle A. Rectangle $AB''MC'$ has dimensions equal to BB' and CC'', and is the sum of two rectangles, $AB''BB'$ and $B'BMC'$. The first of these rectangles is equal in area to triangle BAE, which has AB' for altitude and $BE = 2\ BB'$ for base. The second is equal in area to triangle BEC, which has $NC = B'C'$ for altitude and BE for base. Thus rectangle $AB''MC'$ is equal in area to triangle ABC. Similarly, rectangle $AC''NB'$ is equal in area to triangle ABC, for this rectangle is the difference of rectangles

* Translated from the French by Howard Eves.

$AC''CC'$ and $C'CNB'$, respectively equivalent to triangles CAF and CBF, the difference of which is triangle ABC. We thus have

$$BB' \cdot CC'' \cdot CC' \cdot BB'' = (\text{area } ABC)^2. \tag{1}$$

Now, the hyperbola ($\mathfrak{H}$) which has foci at B and C and which passes through A, has for tangent at A the interior bisector AD of the angle formed by the radii vectors AB and AC, and the ellipse ($\mathcal{E}$) having foci at B and C and passing through A has for tangent at A the exterior bisector of angle A. Therefore*

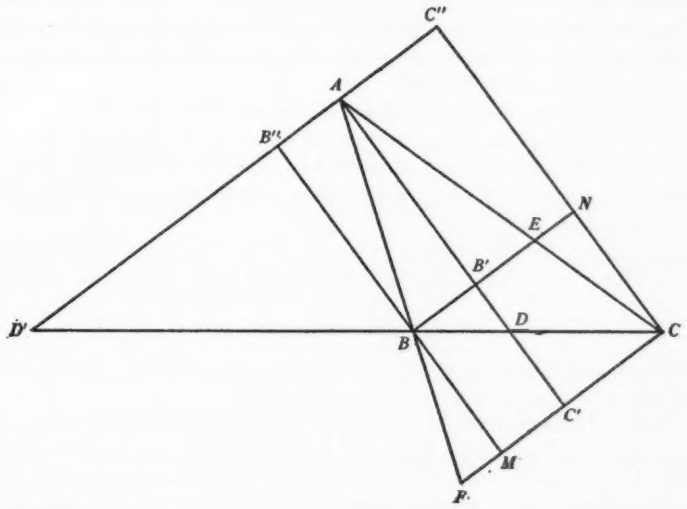
$$\begin{aligned} BB' \cdot CC' &= \tfrac{1}{4}[a^2 - (b-c)^2] \\ &= \tfrac{1}{4}(a+b-c)(a-b+c) = (p-c)(p-b), \end{aligned} \tag{2}$$

$$\begin{aligned} BB'' \cdot CC'' &= \tfrac{1}{4}[(b+c)^2 - a^2] \\ &= \tfrac{1}{4}(b+c+a)(b+c-a) = p(p-a), \end{aligned} \tag{3}$$

whence, by virtue of (1),

$$(\text{area } ABC)^2 = p(p-a)(p-b)(p-c).$$

The consideration of the conics ($\mathfrak{H}$) and ($\mathcal{E}$) remarkably simplifies the calculation of the products (2) and (3), which can also be obtained directly by the evaluation of the segments BB', BB'', CC', CC'' as functions of the sides a, b, c of triangle ABC.



* *The product of the focal perpendiculars on any tangent to a central conic, $x^2/\alpha^2 \pm y^2/\beta^2 = 1$, is constant and equal to β^2.* See almost any analytical geometry text, *e.g.*, Fine and Thompson, p. 85. A synthetic demonstration may be found in Macaulay's Geometrical Conics (2nd ed.), p. 112. (H. Eves)

CLUBS AND ALLIED ACTIVITIES

EDITED BY J. S. FRAME

Send reports of all activities, such as club reports, special features, topics with references, student papers, and other material of interest to J. S. Frame, Michigan State College, East Lansing, Michigan.

CLUB REPORTS 1944–45

Delta-x, University of the City of Toledo

In the past year, *Delta-x* has maintained a membership of almost ninety despite the large number of members called to the armed forces. Papers of interest to students of mathematics were presented at each of the monthly meetings:

The trisection of angles, by Lois Martin

Greek contributions to mathematics, by Al Philop

A comparative history of mathematics, by Dorothy Wehde and Robert Dancer

Graphic solution of equations, by Richard Reisbach

The dozen system, by Virginia Webber

Mathematics used in production, by Sam Kory

The annual Get-Acquainted Roast opened the social activities of the club early in October. A Hard Times Party was held in November and a Christmas party was combined with the regular December meeting. A pot luck supper took place before the March business meeting. The annual *Delta-x* banquet was held in May followed by a talk entitled:

Solution of diophantine equations, by Dr. Lionel Fleischmann.

Delta-x yearbooks were presented to members at the banquet. Lois Martin was the editor. The club voted to place two concrete benches in front of the University as a memorial to its members in the armed services. The last function of the club was a picnic.

Officers elected for the coming year are: President, Albertine Krohn; Vice-President, Dorothy Wehde; Secretary-Treasurer, Lois Zeigler.

Pi Mu Epsilon, Duke University

Four regular meetings were held during the year 1944–45. At the fall initiation meeting, a talk was given on

The theory of ballistics, by Mr. John Kelly of the Aberdeen Proving Ground.

A tea dance and reception was held before Christmas to welcome the new members. At the spring initiation meeting the membership reached 153. Preceding the initiation the talk was given on

Prime numbers, by Professor Leonard Carlitz.

Election of the following officers for 1944–45 took place at the final meeting: President, Jacqueline Quinn; Vice-President, Jane Ammerman; Secretary, Ethelyn Smith; Treasurer, Joseph King.

Mathematics Club, Case School of Applied Science

At the present time the Mathematics Club has about 25 members who regularly attend the meetings and lectures held under the auspices of this group. Membership is open to all interested students of the college. During the past school year, 1944–45, the following talks were given:

Geometry of the complex plane, by George Springer

Fun with figures, by Gerald Levy

Congruences and congruent numbers, by Robert Smith

The algebra of plane and point sets, by Murray Ellis

Special methods for the use of infinite series in the solution of differential equations, an original paper by Professor Jesse Pierce

Approximate numerical solutions to algebraic equations of degree greater than three, by Robert Gall

Some geometrical constructions and impossibility proofs, by Donald Glaser

Introduction to the algebra of matrices, by George Springer

Some elementary deductions from the special theory of relativity by David Dutton.

In addition to these formal talks which were usually followed by general discussion and comment, several meetings of the club were given over to the discussion of special mathematical problems and their solutions. Many solutions to problems appearing in this MONTHLY were presented at these meetings.

For the past year the officers of the club have been: President, Robert Smith; Vice-President, Donald Glaser; Secretary, Robert Gall; and Sponsor, Professor Max Morris.

Pi Mu Epsilon, Hunter College

The topic discussed in the twelve papers which were presented at eight program meetings was:

The theory of functions of a complex variable, presented by the following members: Dorothy Geddes, Theo Gelbfarb, Gladys Heinlein, Marilyn Hochberg, Grace Lesser, Ann Muzyka, Hannahruth Moses, Leila Rubashkin, Joyce Rubin, Wanda Seglow, Marilynn Spanglet, and Alice Winzer.

Twenty-one students and one faculty member were elected to the chapter during the year. The main social event was the festive Twentieth Anniversary Dinner of the chapter, attended by 125 members at the Hotel Shelton, April 28, 1945, and presided over by Professor Jewell Hughes Bushey, Chairman. The address was entitled

Mathematics in everyday life, by the guest speaker, Professor Oystein Ore of Yale University.

Officers for 1944–45 were: President, Dorothy Geddes; Recording Secretary, Marilynn Spanglet; Corresponding Secretary, Rae Adelson; Treasurer, Mae Perlstein. Officers elected for 1945–46 are: President, Mary Greene; Recording Secretary, Carol Podell; Corresponding Secretary, Sally Rothstein; Treasurer, Leila Rubashkin; Director, Dr. Annita Tuller.

Mathematics Club, Hunter College

The Mathematics Club of Hunter College has had a very active year. The club was fortunate in having three very interesting talks on mathematics:

The theory of games, by Dr. Irving Kaplansky of Columbia University

Order of infinities, by Mr. Ernst Straus, Assistant to Professor Einstein

Mathematics and the social sciences, by Professor Jewell Hughes Bushey, Chairman of the Mathematics Department.

Student reports included:

Mathematics and music
Short cuts in mathematics
The history of measurements
The metric system
The history of calculus
Astronomy.

The club made two trips to the mathematics library of Columbia University, once to see the mathematical instruments and once to visit the Plimpton Library. Other activities of the year included learning how to use the slide rule and the solving of mathematical puzzles. Social activities consisted of a Christmas party and a Spring party. A hike is planned for early in September.

The officers of the club were: President, Carol Podell; Vice-President, Sally Rothstein; Secretary, Wilhelmina Fluhr; Treasurer, Paulette Gross; Publicity Manager, Dorothy Beck. Miss Isabel C. McLaughlin was the Faculty Adviser.

Mathematics Club, University of Wisconsin

In view of a wartime scarcity of advanced graduate students, the various mathematics clubs of the University of Wisconsin were merged into one during the academic year 1944–45, and speakers were asked to keep their talks at a fairly elementary level. This expedient proved happy indeed, and it was found worthwhile to hold meetings at intervals of about two weeks. Except where indicated below, the speakers were graduate students or members of the Mathematics Department. The following papers were presented:

A mathematical theory of traffic light signals, by Professor L. R. Wilcox, of the Illinois Institute of Technology

Functions of random variables, by Professor H. P. Evans

Hamilton and his quaternions, by Professor C. C. MacDuffee

Some properties of orthogonal matrices, by Mr. N. A. Wiegmann

Gibb's phenomenon, by Mr. W. G. Scobert

Current problems, by Professor C. J. Everett

Hamilton's principle, by Miss Gloria Olive

How the Fourier series was not discovered, by Professor R. E. Langer

How the Fourier series was discovered, by Professor R. E. Langer

A theorem of Markhoff, by Mr. L. H. Kanter

Applications of the theory of elasticity to plywood, by Professor H. W. March (on leave at Forest Products Laboratory)

The cluster, a generalization of the ring, by Mr. R. A. Good.

Pi Mu Epsilon, University of California at Berkeley

The California Beta chapter of *Pi Mu Epsilon* has been holding regular meetings during the past year and a quarter, at which many interesting and outstanding papers have been presented, as follows:

Higher curves, illustrated by Pollack models, presented in March 1944 by Mr. R. Wakerling

How to turn a line around, by Professor R. M. Robinson

Mathematical quotations from non-mathematical literature, by Professor Max Zorn of UCLA

Orthogonal polynomials, by A. Horn, former teaching assistant in the Mathematics Department

Surface and capacity of the ellipsoid, by Professor D. H. Lehmer

Competition in economics, by Professor G. C. Evans, Head of the Mathematics Department

Exterior ballistics, by Professor L. Swinford

Problems of runs, by Mr. W. M. Chen

Problems concerning the teaching of mathematics, by Mrs. S. L. McDonald

Just how big should a rocket be? by Dr. Sam Schaaf

Ideal numbers, by Dr. L. Walton

Restricted systems of equations, by Professor A. R. Williams

Finite algebras, by Bjarni Jonsson.

The year's activities were concluded at an initiation banquet, at which Allan Davis presided as toastmaster.

Officers for 1944–45 were: Director, Frank Massey; Vice-Director, Bjarni Jonsson; Secretary-Treasurer, Betty Shapiro; Librarian, Sarah Hallam. Officers for 1945–46 are: Director, Bjarni Jonsson; Vice-Director, Betty Shapiro; Secretary, Esther Bateman; Treasurer, Frank Davis; Librarian, Sarah Hallam.

Kappa Mu Epsilon, Central Michigan College of Education

The activities for the year consisted of eight monthly meetings, including two formal initiation programs and the annual spring picnic. Thirty-one new members were initiated on January 10, and thirty-six on May 2. Attendance was greatly increased by the presence of a Navy V-12 unit on campus. Papers presented during the year were as follows:

The value and determination of pi, by Betty Sack

Aeronautical navigation, by Mr. Lester Serier

Mathematical puzzles, by Pauline Nelson.

The fraternity gave donations to the American National Red Cross, and to the War Student Service Fund. Members are now working on a pamphlet for secondary schools on the mathematics courses students should have in high school in preparation for various vocations, such as engineering, law, medicine, pharmacy, nursing, laboratory technicians, and flying. Joyce Sherwood, graduating president, was given a membership in the National Council of the Teachers of Mathematics.

The officers for the year were: President, Joyce Sherwood; Vice-President, Pauline Nelson; Secretary, Emma May Skinner; Treasurer, Frances Teel; and Corresponding Secretary, Mr. L. C. Gray. Summer officers elected at the May meeting were: President, Wayne Munger; Vice-President, P. T. Austin; Secretary, Sol Jacobson; Treasurer, Mike Schweinburg; and Corresponding Secretary, Miss Nikoline Bye. The officers for the year 1945–46 are: President, Margaret Ketchum; Vice-President, Frances Teel; Secretary, Dorothy Michener; Treasurer, Dorothy Sharard; and Corresponding Sectetary, Mr. Lester Serier.

RECENT PUBLICATIONS

EDITED BY H. P. EVANS, University of Wisconsin

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, 531 West 116th Street, New York 27, N. Y., and not to any of the other editors or officers of the Association.

Elementary Statistics. By Hyman Levy and E. E. Preidel. New York, The Ronald Press Company, 1945. 7+184 pages. $2.25.

If any one is so benighted as to be still of the opinion, which apparently had some credence at one time, that anything can be proved by statistics, he will be enlightened by the first paragraph of the introduction to this little book. The authors point out that while it is, of course, possible to fake or doctor evidence, statistics properly used can detect and refute such evidence.

Many books on elementary statistics merely give methods of applying, in a mechanical manner, various statistical operations. This book is not of that type. It gives, by means of good discussions and simple examples, an insight into the meaning and purpose of fundamental statistical concepts, and the reader who carefully studies it will be able to make elementary statistical analyses and inferences in an intelligent fashion. A few exercises are provided at the ends of chapters and elsewhere.

The book deals with classic statistics and contains nothing, for example, about Student's distribution, chi square, or Fisher's analysis of variance. However, it does emphasize the importance of small samples. Also it contains a chapter on quality control.

It covers more material than the number of pages might indicate, since the print is small. This material is part of the work on statistics given at the Imperial College of Science, London, where the authors are located, to students of mathematics, physics, and biology during the past twenty years. Recently it has been added to the curriculum of first-year engineering students at that institution.

P. R. RIDER

Plane and Spherical Trigonometry. By F. M. Morgan. New York, American Book Company, 1945. 5+243+72 pages. $2.50.

As is stated in the preface "this brief presentation of Plane and Spherical Trigonometry emphasizes the numerical aspect and gives as much theory as is necessary for a thorough preparation for further work in mathematics." The book has been written with great care and considerable pedagogical insight. The numerous applications to mechanics and navigation will help to motivate the student to the study of trigonometry by impressing upon him the timeliness of the subject. The occasional historical notes will give him some knowledge of its development. The text as well as the problem lists contain many thought provoking questions. The problem lists contain tests of various types: drill exercises, true and false tests, completion tests and mastery tests.

In addition to the material commonly found in trigonometry texts, this book contains a chapter on DeMoivre's theorem, a section on sailing, and the Mercator map. The trigonometric functions of an acute angle are first defined, and sufficient applications to the solutions of the right triangle are given to familiarize the student before the functions of the general angle are introduced. The idea of the polar triangle is used to reduce the number of cases for the solution of the oblique spherical triangle. The Appendix contains a chapter on logarithms. Five place tables of logarithms of numbers, of logarithms of trigonometric functions, of logarithmic and natural haversines, and of natural trigonometric functions are included in the book. No list of answers to the problems is given.

The entire book is remarkably well written and the reviewer recommends it highly for use in high schools and colleges. The type and presswork are excellent.

H. P. THIELMAN

NEW BOOKS RECEIVED

Applied Mathematics for Radio and Communication Engineers. By C. E. Smith. New York and London, McGraw-Hill Book Co., Inc., 1945. 10+336 pages. $3.50.

An Historical and Analytical Bibliography of the Literature of Cryptology. By J. S. Galland. Evanston, Northwestern University, 1945. 8+209 pages. $5.00.

Hydrodynamics. By Horace Lamb. (6th edition; reprint). New York, Dover Publication, 1945. 15+738 pages. $4.95.

An Introduction to Mathematics for Teachers. By L. E. Boyer. New York, Henry Holt and Co., 1945. 17+478 pages. $3.25.

Note by the editor. A recent review of *The Education of T. C. Mits* (this MONTHLY, vol. 52, 1945, p. 270) might lead the reader to infer that the book was written by H. G. Lieber, with L. R. Lieber supplying the drawings. The purpose of the present note is to correct this impression; the text is due to L. R. Lieber and the drawings are by H. G. Lieber. H. P. E.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, ORRIN FRINK, JR., AND HOWARD EVES

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to Howard Eves, College of Puget Sound, Tacoma 6, Washington.

The department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 691. *Proposed by E. K. Paxton, Washington and Lee University*

The top of a certain bridge is 215 feet above the ground and 100 feet wide. Neglecting air resistance, find the minimum initial velocity with which an arrow is shot in order to go over the bridge. At what distance from the base must the archer stand? What is the angle of elevation? How high does the arrow go? (Assume the arrow leaves the bow at a height of 5 feet above the ground.)

E 692. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider $2p+1$ consecutive integers. Show that: (1) the product of the first p of them diminished by the product of the last p gives an integer divisible by $p(p+1)$; (2) the product of the first p increased by the product of the last p gives an integer divisible by $p(p+1)$, unless $p+1$ is a prime factor of the remaining integer of the set.

(This is a generalization of problem 203 by Fitz-Patrick, p. 32 of G. de Longchamps' *Exercises d'arithmétique.*)

E 693. *Proposed by N. A. Court, University of Oklahoma*

Through a given point draw a line meeting three given planes in three points so that the anharmonic ratio of the four points shall have a given value.

E 694. *Proposed by J. D. Bankier, Sedbergh School, Montebello, Province of Quebec*

A sequence $\{x_n\}$ is defined recursively, in terms of two numbers x_0 and x_1, by the formula

$$x_n = \frac{n-1}{n} x_{n-1} + \frac{1}{n} x_{n-2}.$$

To what value does the sequence converge?

E 695. *Proposed by H. L. Lee, University of Tennessee*

Find triangles whose sides are integers in arithmetic progression, and whose areas are integers.

SOLUTIONS

A Generalization of E 255

E 657 [1945, 95]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the locus of centers of spheres passing through two given points and touching a given sphere.

Solution by Howard Eves, College of Puget Sound. Let O be the center and r the radius of the given sphere, let A and B be the two given points, $\angle OAB=\theta$, and let P be a point on the required locus. If the value of r lies between OA and OB, we easily see that there is no real locus. So we consider in turn the two other possibilities.

(a) Suppose both OA and OB are greater than r. Then we must have $|PO-PA|=r$, and one surface on which P lies is a hyperboloid of revolution of two sheets with foci O and A. The rotation angle, ϕ, of the asymptotic cone of the hyperboloid, is given by $\cos\phi=r/OA$. But P lies also on the plane perpendicularly bisecting the segment AB. Therefore the locus of P is

(1) a circle if $\theta=0$ or π,

(2) an ellipse if $0<\theta<\frac{1}{2}\pi-\phi$ or $\frac{1}{2}\pi+\phi<\theta<\pi$,

(3) a parabola if $\theta=\frac{1}{2}\pi\pm\phi$,

(4) a hyperbola if $\frac{1}{2}\pi-\phi<\theta<\frac{1}{2}\pi+\phi$.

(b) Suppose both OA and OB are less than r. Here $PO+PA=r$, and the locus of P is the intersection of an ellipsoid of revolution and a plane, *viz.*,

(1) a circle if $\theta=0$ or π,

(2) an ellipse otherwise.

Also solved by J. M. Feld, H. L. Lee, P. D. Thomas, J. T. Webster, and the proposer. N. A. Court draws attention to his *Modern Pure Solid Geometry* p. 196, art. 614.

The Conformal Points of a Projection

E 661 [1945, 159]. *Proposed by Howard Eves, College of Puget Sound*

A plane p is projected from a point L onto a plane p'. Find those points on p for which all angles on p having such a point for vertex are invariant under the projection.

Solution by the Proposer. We adopt the convention that angular directions on p (or p') are positive if they are counterclockwise when p (or p') is viewed from point L. We now make the

DEFINITIONS. A point on p will be called a *positive isocenter* or a *negative isocenter* on p according as all the angles on p having the point for vertex project respectively into equal and similarly directed or equal and oppositely directed angles on p'.

By an elementary synthetic treatment we shall show that when p and p' are nonparallel there is one and only one positive isocenter and one and only one negative isocenter on p. We shall also geometrically locate these isocenters.

THEOREM 1. *If P is a positive (or negative) isocenter on p, then its image, P', is a positive (or negative) isocenter on p'.*

THEOREM 2. *If p and p' are parallel, then every point on p is a positive or a negative isocenter according as L does not or does lie between p and p'.*

THEOREM 3. *If p and p' are not parallel, then p cannot have two positive isocenters.*

For suppose P and Q are two positive isocenters on p. Draw any circle s on p passing through P and Q, and let R be any other point on s. Then, since $\angle QPR=\angle Q'P'R'$ and $\angle PQR=\angle P'Q'R'$, it follows that $\angle PRQ=\angle P'R'Q'$, whence s projects into a circle s'. Thus all circles on p through P and Q project into circles on p' through P' and Q'. But this is impossible. For consider a circle on p, passing through P and Q and cutting the vanishing line on p. This circle must project into a hyperbola. Thus the original assumption of the existence of two positive isocenters is incorrect.

THEOREM 4. *Let p and p' be nonparallel and let O be the foot of the perpendicular from L onto p, and let V on p be the image of V', the foot of the perpendicular from L onto p'. Then any line on p perpendicular to VO is parallel to any line on p' perpendicular to $V'O'$.*

Draw VQ on p perpendicular to VO. Then

$$LQ^2 = LO^2 + OQ^2 = LO^2 + VO^2 + VQ^2 = LV^2 + VQ^2.$$

Therefore VQ is perpendicular to LV, and therefore to plane VLO. Hence any line on p perpendicular to VO is perpendicular to plane VLO. Similarly, any line on p' perpendicular to $O'V'$ is perpendicular to plane VLO. This proves the theorem.

THEOREM 5. *If p and p' are nonparallel, then p has exactly one positive isocenter, and it is the intersection with p of the bisector of the angle formed by the perpendiculars dropped from L onto p and p'.*

We adopt the notation of theorem 4. Let P be any point on p and let W be the foot of the perpendicular from P on VO. Then W' is the foot of the perpendicular from P' on $V'O'$ and, by Theorem 4, PW is parallel to $P'W'$. Let I be the intersection with VO of the bisector of angle VLO. Draw WK perpendicular to LV to cut LV in K and LI in J. Then

$$IW/I'W' = JW/I'W' = LW/LW' = WP/W'P'.$$

Hence $\angle WIP=\angle W'I'P'$. Similarly, if Q is any other point on p, $\angle QIW=\angle Q'I'W'$. Hence $\angle QIP=\angle Q'I'P'$, and I is a positive isocenter on p. By theorem 3, I is the only positive isocenter on p.

THEOREM 6. *If p and p' are nonparallel, then p has exactly one negative isocenter, and it is the intersection with p of the external bisector of the angle formed by the perpendiculars dropped from L on p and p'.*

We may give a proof of this theorem analogous to that given for theorem 5.

Note. The positive isocenter has long been known and employed in photogrammetry (the science of surveying by means of photographs), but the negative isocenter seems to have escaped notice. For an analytical treatment of the above and for an application to photogrammetry see "Analytical and Graphical Rectification of a Tilted Photograph," by Howard Eves, *Photogrammetric Engineering*, April-May-June issue, 1945.

Two Scales of Notation

E 662 [1945, 159]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

A number is represented by a in the scale of α and by b in the scale of β ($\beta<\alpha$). Regarding both a and b as written in the scale of α, we write the difference $b-a=c$. Show how to determine the greatest possible value of a for given values of α, β, c; *e.g.*, when $\alpha=10, \beta=7$, and $c=3501$.

Solution by R. C. Buck, Harvard University. We have

$$N = \sum a_k\alpha^k = \sum b_k\beta^k$$

and

$$c = \sum b_k\alpha^k - \sum a_k\alpha^k = \sum b_k(\alpha^k - \beta^k).$$

Set $\gamma_k=\alpha^k-\beta^k$, $k=0, 1, 2, \cdots$. If c is expanded in terms of the γ_k, choosing always the greatest values for the coefficients of $\gamma_n, \gamma_{n-1}, \cdots$, in order, then the maximum a will be found. Thus

$$b_n = [c/\gamma_n],\ b_{n-1} = [(\text{remainder } c/\gamma_n)/\gamma_{n-1}], \cdots, b_0 = \beta - 1.$$

In the special case where $\alpha=10, \beta=7, c=3501$ we find

$$\{\gamma_k\} = \{0, 3, 51, 657\}, \qquad \{\beta_k\} = \{6, 4, 4, 5\}.$$

Therefore

$$a = 5(7^3) + 4(7^2) + 4(7) + 6(1) = 1945.$$

Also solved by Murray Barbour, D. H. Browne, F. M. Carpenter, Robert Hoskins, E. E. Osborne, E. D. Schell, E. P. Starke, and the proposer

Two Powers Differing by Unity

E 663 [1945, 159]. *Proposed by Irving Kaplansky, Columbia University*

If $2^n+1=p^r$, where p is a prime, prove that r is a power of 2 (including the possibility $r=2^0=1$).

I. *Solution by Joseph Rosenbaum, Bloomfield, Connecticut.* The supposition that $r=hk$, where k is odd and greater than 1 gives

$$(p^k - 1)[(p^k)^{k-1} + (p^k)^{k-2} + \cdots + p^k + 1] = 2^n.$$

This is impossible because the bracket is also odd and greater than 1. Hence $r = 2^m$.

However, when $r = 2^m$, the given equation can be written as

$$(p^{2^{m-1}} - 1)(p^{2^{m-1}} + 1) = 2^n,$$

and it is easily proved that this is possible only when $m = 0$ or 1, and that for the latter situation $p = 3$.

It is also seen that the hypothesis that p is a prime is superfluous.

II. *Solution by D. W. Alling, Rochester, New York.* We have $p^r \equiv 1 \pmod{2^n}$, and it is clear that p belongs to the exponent r mod 2^n. But $\phi(2^n) = 2^{n-1}$ and $r \mid 2^{n-1}$. Hence r is a power of 2. If $r > 1$, then r is even and the integral factorization

$$(p^{r/2} - 1)(p^{r/2} + 1) = 2^n$$

is possible. Therefore

$$p^{r/2} - 1 = 2^a, \qquad p^{r/2} + 1 = 2^b, \qquad 2^a - 2^b = -2.$$

Solving we find, uniquely, $b = 2$, $a = 1$, $n = 1$.

Also solved by D. W. Alling (another solution), Murray Barbour, D. H. Browne, R. C. Buck, A. Charnes, Roy Dubisch, Paul Erdős, N. J. Fine, J. B. Kelly, E. D. Schell, Peter Scherk (two ways), E. P. Starke, and the proposer (two ways).

The solutions of Murray Barbour, Roy Dubisch, N. J. Fine, J. B. Kelly, E. D. Schell, and one of the solutions of the proposer are essentially like solution I above. The alternate solution of D. W. Alling and the solution of D. H. Browne utilize the fact that all primes are of the form $4k \pm 1$. The proposer offered a second solution using the theory of Galois groups. R. C. Buck established the more general theorem: If $q^n + 1 = a^r$, q *prime, then* $r = 1$ *except for the special case* $2^3 + 1 = 3^2$. *If, further, a is prime, then the only solutions are the Fermat primes,* $p = 2^{2^n} + 1$. Peter Scherk and Paul Erdös solved essentially Buck's extension. Most solvers noted the uniqueness of solution when $r > 1$, and several observed that p need not be restricted to a prime.

An Integral Related to the Gamma Function

E 664 [1945, 159]. *Proposed by D. H. Browne, Buffalo, N. Y.*

Prove that if $x < 1$,

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} \int_1^{\infty} t^n e^{-t} dt = \frac{e^{x-1}}{1-x}.$$

I. *Solution by N. J. Fine, Purdue University.* The problem should be stated with the condition $|x| < 1$. Set $B_n = 1/n! \int_1^{\infty} t^n e^{-t} dt$. $B_0 = e^{-1}$ and an integration by by parts shows that $B_n = e^{-1}/n! + B_{n-1}$, so $B_n = e^{-1} \sum_{k=0}^{n} (1/k!)$. Hence, if $|x| < 1$,

$$\sum_{n=0}^{\infty} B_n x^n = e^{-1}\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{x^n}{k!} = e^{-1}\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{n=k}^{\infty}x^n = e^{-1}\sum_{k=0}^{\infty}\frac{x^k}{k!}\cdot\frac{1}{1-x} = \frac{e^{x-1}}{1-x}.$$

II. *Solution by Harley Flanders, University of Chicago.* The problem should be stated with the condition $|x|<1$. Set $c_n=\int_1^\infty t^n e^{-t}dt$. Then we have

$$n! = \int_0^\infty t^n e^{-t}dt > c_n = \int_0^\infty t^n e^{-t}dt - \int_0^1 t^n e^{-t}dt > n! - \int_0^1 e^{-t}dt = n! - 1 + e^{-1}.$$

Therefore

$$\sum_{n=0}^{\infty} x^n > \sum_{n=0}^{\infty}(c_n x^n)/n! > \sum_{n=0}^{\infty}[1-(1-e^{-1})/n!]x^n.$$

By the "ratio test" we see that both extreme series, and hence the given series, converge in the interval $|x|<1$. Hence the given series converges uniformly in that interval by a known theorem on power series and we may interchange the operations:

$$\sum_{n=0}^{\infty}\frac{x^n}{n!}\int_1^\infty t^n e^{-t}dt = \int_1^\infty\left[\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right]e^{-t}dt = \int_1^\infty e^{(x-1)t}dt = \frac{e^{x-1}}{1-x},$$

(since $x-1<0$).

Also solved by D. W. Alling, Murray Barbour, Ellen Buck, Sidney Glusman, J. E. Hanson, J. B. Kelly, E. E. Osborne, E. P. Starke, C. W. Topp, and J. T. Webster.

J. B. Kelly obtained his solution from results of E 654. E. P. Starke showed, as an incidental note to his solution, that

$$ec_n/n! < e < (ec_n+1)/n!,$$

$$ec_1 = 2, \qquad ec_{n+1} = 1 + (n+1)ec_n.$$

Since the integers ec_n are easy to compute we then have here a novel way of computing the numerical value of e.

Circles Covering a Given Curve

E 665 [1945, 159]. *Proposed by L. A. Santalo, Rosario, Argentina*

Let C be a closed convex plane curve with continuous radius of curvature R. Let R_M be the greatest value of R. Given $\lambda \geqq R_M$, show that the area F_λ covered by the centers of circles of radius λ which contain C in their interior is given by

$$F_\lambda = F - L\lambda + \pi\lambda^2,$$

where L and F are the length and area of C.

Solution by R. A. Rosenbaum, U.S.N.R. Problem E 630 [1945, 160] can be easily generalized so as to include the present problem. The generalization of E 630 is:

For any point P of a given closed convex curve C, let P' be that point on the normal to C at P for which $PP'=k$, a constant, taken as positive or negative according as P' lies on the exterior or the interior normal. The locus of P' is a curve C'. Let s, s' be the respective lengths of C, C', and A, A' the areas of these curves. Then

$$s' = s + 2\pi k,$$

$$A' = A + sk + \pi k^2.$$

The proof as given for E 630 applies to this generalization, and the present problem is seen to be a special case.

Also solved by R. H. Wilson, Jr. and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis 5, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known text-books or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

4176. *Proposed by H. S. M. Coxeter, University of Toronto*

Prove the following two theorems in affine geometry of three dimensions:

(a) If all the faces of a convex polyhedron are parallelograms, their number is the product of two consecutive integers;

(b) If each face of a convex polyhedron has a center of symmetry, the whole polyhedron has a center of symmetry.

4177. *Proposed by P. R. Halmos, Syracuse University*

What is the smallest number of people sufficient to ensure that the probability that there be at least two with the same birthday is at least 1/2? It is to be assumed that any two days of the year are equally likely to be the birthday of a given individual and that there are no leap years.

4178. *Proposed by N. A. Court, University of Oklahoma*

If among the twelve points of intersection of a quadric surface with the edges of a tetrahedron there are six points located on the six edges, such that the three lines joining the points on opposite edges are concurrent, the remaining six points of intersection also have this property.

Note. This is a converse of a known proposition (see, for inst. Baker, *Principles of geometry*, vol. 3, p. 54, ex. 17, 1923): The transversal is drawn from a

point O to each pair of opposite joins of four points A, B, C, D, so giving rise to six points. Six other points, one on each join, are similarly obtained from another point O'. The twelve points lie on a quadric.

4179. *Proposed by J. R. Musselman, Western Reserve University*

The poles of the medians of a triangle $A_1A_2A_3$ as to its circumcircle are three points of a line; those points where the external bisectors of the angles of the tangential triangle of $A_1A_2A_3$ meet the opposite sides of this tangential triangle.

4180. *Proposed by Victor Thébault, Tennie, Sarthe, France*

The product of k positive integers whose sum is N, where $N=kp+h$, is a maximum when h of the factors are equal to $p+1$ and the $k-h$ others to p.

Dedicated to E. P. Starke.

SOLUTIONS

Trigonometric Determinants

4125 [1944, 352]. *Proposed by Hüseyin Demir, Columbia University*

Prove that

$$\begin{vmatrix} \sin\theta_1 & -e^{-i\theta_1} & 0 & 0 & \cdots & 0 & 0 \\ \sin\theta_2 & e^{i\theta_2} & -e^{-i\theta_2} & 0 & \cdots & 0 & 0 \\ \sin\theta_3 & 0 & e^{i\theta_2} & -e^{-i\theta_3} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \sin\theta_n & 0 & 0 & 0 & \cdots & 0 & e^{i\theta_n} \end{vmatrix} = \sin(\theta_1+\theta_2+\cdots+\theta_n).$$

Solution by Mary L. Boas, Tufts College. Put each $\sin\theta=(e^{i\theta}-e^{-i\theta})/2i$ and remove the factor $1/2i$ outside the determinant. Subtract from each element of the first column the sum of all the other elements in its row. The determinant then becomes

$$\frac{1}{2i}\begin{vmatrix} e^{i\theta_1} & -e^{-i\theta_1} & 0 & \cdots & 0 & 0 \\ 0 & e^{i\theta_2} & -e^{-i\theta_2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & e^{i\theta_{n-1}} & -e^{-i\theta_{n-1}} \\ -e^{-i\theta_n} & 0 & 0 & \cdots & 0 & e^{i\theta_n} \end{vmatrix}.$$

Expand by elements of the first column. The minor of $e^{i\theta_1}$ is $e^{i(\theta_2+\theta_3+\cdots+\theta_n)}$ since all elements below the main diagonal of this minor are zero. The minor of $-e^{-i\theta_n}$ is $(-1)^{n-1}e^{-i(\theta_1+\theta_2+\cdots+\theta_{n-1})}$ since all elements above its main diagonal are zero. Therefore the determinant equals

$$\frac{1}{2i}\left[e^{i(\theta_1+\theta_2+\cdots+\theta_n)}-e^{-i(\theta_1+\theta_2+\cdots+\theta_n)}\right]=\sin(\theta_1+\theta_2+\cdots+\theta_n).$$

Solved also by E. F. Allen, Murray Barbour, C. B. Barker, Jr., Shepard Bartnoff, R. P. Boas, Jr., Mrs. R. C. Buck, Howard Eves, Clifford Gardner, P. C. Hammer, R. Hamming, J. F. Hofmann, L. M. Kelly, E. Lukacs, Norman Miller, Henry Nelson, Ivan Niven, H. N. Shapiro, Robert Steinberg, R. H. Wilson, Jr., and the proposer.

Editorial Note. About half of the solutions used induction proofs and about the same number used simple determinant transformations without induction. Hammer considered the transformation of the determinant and its value by replacing θ_j by $\pi/2-\theta_j$ which gives after reduction a determinant with $a_{j1}=\cos\theta_j$ in the first column and the principal diagonal $\cos\theta_1, e^{-i\theta_2}, e^{-i\theta_3}, \cdots$ and the parallel above it $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, \cdots$ with zeros in the remaining places. He found for the value of the determinant $\cos\sum\theta_j$ if n is odd, and $-i\sin\sum\theta_j$ if n is even. A simpler procedure is to make the same change in the first column but to alter the principal diagonal to $\cos\theta_1, -e^{i\theta_2}, -e^{i\theta_3}, \cdots, -e^{i\theta_n}$ and leave the rest of the original determinant unaltered. The value of this determinant is the same as that for Hammer's determinant.

Spherical Loxodromes

4128 [1944, 409]. *Proposed by C. E. Springer, University of Oklahoma*

Consider the tangent planes to a sphere at three points A, B, C of a curve lying on the sphere. Let R be the limiting point of intersection of the planes as B and C move independently along the curve and approach coincidence with A. Each curve on the sphere through A has its corresponding R point. Prove that the curves through A, the locus of whose R points is a certain straight line lying in the tangent plane to the sphere at A, are the loxodromes through A.

Solution by the Proposer. The tangent plane at the point $A(u, v)$ on the sphere represented by

$$x = \sin u\cos v, \qquad y = \sin u\sin v, \qquad z = \cos u \tag{1}$$

has the equation

$$x\cos v + y\sin v + z\cot u = \csc u. \tag{2}$$

The radius of the sphere is taken as unity for convenience, without loss of generality. If the curve through A is represented by $u=u(s)$, $v=v(s)$, (the parameter s denoting length of arc), then the coordinates of its "R" point satisfy equation (2), together with equations

$$x\sin v\cdot v' - y\cos v\cdot v' + z\csc^2 u\cdot u' = \csc u\cot u\cdot u' \tag{3}$$

and

$$\begin{aligned} &x(\cos v\cdot v'^2 + \sin v\cdot v'') + y(\sin v\cdot v'^2 - \cos v\cdot v'') \\ &\qquad + z(\csc^2 u\cdot u'' - 2\csc^2 u\cot u\cdot u'^2) \\ &\qquad = \csc u[\cot u\cdot u'' - u'^2(\csc^2 u + \cot^2 u)]. \end{aligned} \tag{4}$$

Primes indicate differentiation with respect to s.

In solving equations (2), (3), (4), it is to be noted that the coordinates of R are given by

$$x = -\frac{u'}{v'}\sin v \sec u, \qquad y = \frac{u'}{v'}\cos v \sec u, \qquad z = \sec u, \tag{5}$$

if, and only if, the functions $u(s)$, $v(s)$ satisfy the differential equation

$$u'v'' - u''v' + u'^2v' \cot u = 0. \tag{6}$$

For all values of u'/v' the point R given by (5) lies on the line of intersection of the planes

$$x \cos v + y \sin v = 0, \qquad z = \sec u, \tag{7}$$

and this line lies in the tangent plane to the sphere at A.

Equation (6) can be written in the form

$$\frac{d^2v}{du^2} + \cot u \frac{dv}{du} = 0. \tag{8}$$

The curves whose equations satisfy (8) are given by

$$v = c_1 \log \tan \frac{u}{2} + c_2 \tag{9}$$

where c_1 and c_2 are constants. Equation (9) represents the loxodromes on the sphere. Therefore, the loxodromes through a point on a sphere have the property that the locus of their "R" points is a straight line lying in the tangent plane to the sphere at the point.

Note: The Proposer has published the differential equation defining *dual geodesics* on a general surface in metric space—*Dual Geodesics on a Surface*, Bull. Amer. Math. Soc., vol. 48, No. 12, pp. 901–906, December, 1942. For the unit sphere, this differential equation reduces to equation (8). Therefore, the result of this problem can be stated as follows: The dual geodesics on a sphere are the loxodromes.

Editorial Note. The curve γ lies on the sphere (O) with center O and unit radius, and passes through the points A, B_1, B_2; the plane $[A, B_1, B_2]$ through these three points meet (O) in a circle with center $\overline{C}$, the foot of the perpendicular from O to this plane. In the limit where $B_1, B_2 \to A$ the plane $[A, B_1, B_2] \to \pi$, the osculating plane for γ at A; the circle with center $\overline{C} \to$ the circle of plane curvature for γ at A with center C where OC is perpendicular to π. Let $\mathbf{t}, \mathbf{n}, \mathbf{b}$ be the unit vector tangent, principal normal, and binormal at A, forming a right hand system, then $\pi = [\mathbf{t}, \mathbf{n}]$ and OC is parallel to $\mathbf{b}$.

The tangent planes to (O) at A, B_1, B_2 meet in a point $\overline{C}$, and $\overline{C}$, $\overline{C}$ are inverse points re (O). Hence $\overline{C} \to R$ and C, R are inverse points re (O); C lies on (OA)

the sphere with diameter OA, and this sphere is the inverse of the tangent plane to (O) at A. Now suppose that there is an infinite set of curves γ on (O) passing through A such that there is an infinite set of points R lying on a straight line l in the tangent plane at A, that is they have the R property. Then the plane $[l, O]$ meets (OA) in a fixed circle, the inverse of l, and C lies on this fixed circle and AC moves on a fixed cone with vertex A. For a chosen tangent $\mathbf{t}$ at A, the plane through OA perpendicular to $\mathbf{t}$ meets the fixed circle again in C where the vector $\overrightarrow{AC}=\rho\mathbf{n}$, $\rho=1/\kappa$, $\sigma=1/\tau$, κ, τ being the curvature and torsion for the γ considered. The position of $\mathbf{t}$ in the tangent plane is fixed by the angle α with an initial line defined below. Thus at A, $\mathbf{t}$, $\mathbf{n}$, $\mathbf{b}$, ρ, $\overrightarrow{OC}=\mathbf{c}$ are determined functions of α which enters only in trigonometric form; and two different values of α give different values to each of these five quantities.

The equations (1), (2), (3) in the solution of 4105 [1945, 221] give

$$(1) \qquad (\sigma\rho')^2+\rho^2=1, \qquad (\sigma\rho')'+\tau\rho=0, \qquad \mathbf{c}=-\sigma\rho'\mathbf{b}.$$

Set $\bar{\mathbf{t}}=\mathbf{r}\times\mathbf{t}$, then by the methods of 4105 we get

$$(2) \qquad \mathbf{n}=-\sigma\rho'\bar{\mathbf{t}}-\rho\mathbf{r}, \qquad \mathbf{b}=\rho\bar{\mathbf{t}}-\sigma\rho'\mathbf{r}.$$

With the parametric coordinates u, v in the above solution we have

$$(3) \qquad d\mathbf{r}=\mathbf{r}_1du+\mathbf{r}_2dv, \quad (ds)^2=(du)^2+\sin^2 u(dv)^2, \quad \mathbf{r}_1=\mathbf{t}_1, \quad \mathbf{r}_2=\sin u\mathbf{t}_2,$$

with the origin of vectors at 0 and $\mathbf{t}_1$, $\mathbf{t}_2$, $\mathbf{r}$ form a right-hand orthogonal system of unit vectors. We may write

$$(4) \qquad \begin{gathered}\mathbf{t}=\mathbf{t}_1\cos\alpha+\mathbf{t}_2\sin\alpha, \qquad \bar{\mathbf{t}}=-\mathbf{t}_1\sin\alpha+\mathbf{t}_2\cos\alpha,\\ u'=du/ds=\cos\alpha, \qquad v'=\sin\alpha\csc u, \qquad dv/du=\tan\alpha\csc u.\end{gathered}$$

For any curve on (O) it easily follows from a figure that

$$(5) \qquad \mathbf{r}_{11}=-\mathbf{r}, \qquad \mathbf{r}_{12}=\cot u\mathbf{r}_2, \qquad \mathbf{r}_{22}=-\sin u(\mathbf{r}\sin u+\mathbf{r}_1\cos u).$$

Computation then gives

$$(6) \qquad \kappa\mathbf{n}=-\mathbf{r}+(\alpha'+\sin\alpha\cot u)\bar{\mathbf{t}}, \qquad \kappa\mathbf{b}=(\alpha'+\sin\alpha\cot u)\mathbf{r}+\bar{\mathbf{t}}.$$

In order for the system of curves γ to have the R property at A, the vector $\kappa\mathbf{b}$ with its origin at O must lie in a fixed plane. This requires that α' be a linear homogeneous function of $\sin\alpha$ and $\cos\alpha$, $\alpha'=d_1\sin\alpha+d_2\cos\alpha$, where d_1 and d_2 may be functions of u and v. The fixed plane is perpendicular to the vector

$$(7) \qquad d=(d_1+\cot u)\mathbf{t}_1-d_2\mathbf{t}_2+\mathbf{r}.$$

A system of curves γ which has the R property at each point of a member satisfies the differential equation

$$(8) \qquad \frac{d^2v}{du^2}=d_1\sin^2 u\left(\frac{dv}{du}\right)^3+d_2\sin u\left(\frac{dv}{du}\right)^2+(d_1-\cot u)\frac{dv}{du}+d_2\csc u.$$

The simplest case is where the line l is the line at infinity in the tangent plane; the fixed circle reduces to the point O, and the curves γ are the geodesics, *i.e.*, great circles. Here $d_1 = -\cot u$, $d_2 = 0$. Another special case is where $\alpha' = 0$, $d_1 = d_2 = 0$, so that each curve γ has a constant α at each of its points: hence they are the loxodromes.

A Probability Identity

4130 [1944, 409]. *Proposed by J. R. Musselman, Western Reserve University*

Show that

$$\sum_{j=1}^{n} {}_nC_j \frac{(-1)^{j-1}}{j} = \sum_{j=1}^{n} \frac{1}{j}.$$

I. *Solution by G. B. Lang, University of Florida.* Denote the sum on the left by $f(n)$, then

$$\Delta f(n) = f(n+1) - f(n) = \frac{(-1)^n}{n+1} + \sum_{j=1}^{n} [{}_{n+1}C_j - {}_nC_j] \frac{(-1)^{j-1}}{j}$$

$$= \frac{(-1)^n}{n+1} + \sum_{j=1}^{n} {}_nC_{j-1} \frac{(-1)^{j-1}}{j} = \frac{1}{n+1}\left[(-1)^n - \sum_{j=1}^{n} {}_{n+1}C_j (-1)^j \right]$$

$$= \frac{1}{n+1}[-(-1)^{n+1} - (1-1)^{n+1} + (-1)^{n+1} + 1].$$

Hence $\Delta f(n) = 1/(n+1)$, and summation of each member of this equation from $n=1$ to $n-1$, noting that $f(1) = 1$, gives the desired result.

II. *Solution by G. T. Williams, Student, Harvard University.* Set

$$f(x) = \sum_{\nu=1}^{n} (-1)^{\nu-1} \binom{n}{\nu} \frac{x^\nu}{\nu}$$

then

$$f'(x) = -\frac{1}{x}\left\{ -1 + \sum_{\nu=0}^{n} (-1)^\nu \binom{n}{\nu} x^\nu \right\} = \frac{1-(1-x)^n}{1-(1-x)}$$

$$= \sum_{\nu=1}^{n} (1-x)^{\nu-1}.$$

Then

$$f(x) = -\sum_{\nu=1}^{n} \frac{(1-x)^\nu}{\nu} + c, \qquad f(0) = 0 = -\sum_{\nu=1}^{n} \frac{1}{\nu} + c, \qquad f(1) = c,$$

$$f(1) = \sum_{\nu=1}^{n} \frac{(-1)^{\nu-1}}{\nu} \binom{n}{\nu} = \sum_{\nu=1}^{n} \frac{1}{\nu}.$$

Here is a related inversion formula which may be of interest:

$$g(n) = \sum_{\nu=c}^{n} (-1)^{\nu} \binom{n}{\nu} f(\nu), \qquad n = c, c+1, c+2, \cdots,$$

$$f(n) = \sum_{\nu=c}^{n} (-1)^{\nu} \binom{n}{\nu} g(\nu), \qquad n = c, c+1, c+2, \cdots.$$

The proof depends upon the following Lemma:

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{s\nu}{\mu} = 0, \qquad \text{for } \mu < n,$$

$$= (-1)^{n} s^{n}, \quad \text{for } \mu = n.$$

Proof of the Lemma

$$\{(1+x)^{s} - 1\}^{n} = \left\{ sx + \binom{s}{2} x^{2} + \cdots \right\}^{n} = s^{n} x^{n} + a_{1} x^{n+1} + \cdots$$

$$= \sum_{\nu=0}^{\infty} \binom{n}{\nu} (-1)^{n-\nu} \sum_{\mu=0}^{\infty} \binom{s\nu}{\mu} x^{\mu}$$

$$= \sum_{\mu=0}^{\infty} (-1)^{n} x^{\mu} \sum_{\nu=0}^{\infty} (-1)^{\nu} \binom{n}{\nu} \binom{s\nu}{\mu}.$$

Since n is the lowest power of x the proof is complete. The proof of the inversion formula is as follows, taking $s=1$:

$$\sum_{\nu=c}^{n} (-1)^{\nu} \binom{n}{\nu} \sum_{\mu=c}^{\nu} (-1)^{\nu} \binom{\nu}{\mu} f(\mu) = \sum_{\mu=c}^{n} (-1)^{\mu} f(\mu) \sum_{\nu=\mu}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{\nu}{\mu}$$

$$= (-1)^{n} f(n) \binom{n}{n} \binom{n}{n} (-1)^{n} = f(n).$$

This completes the proof.

A related formula is as follows

$$\sum_{\nu=0}^{n} (-1)^{\nu} \binom{n}{\nu} \binom{2\nu}{\mu} = (-1)^{n} \binom{n}{\mu - n} 2^{2n-\mu}.$$

Proof

$$(x^{2} + 2x)^{n} = x^{n} \sum_{\nu=0}^{n} \binom{n}{\nu} x^{\nu} 2^{n-\nu} = \sum_{\mu=n}^{2n} \binom{n}{\mu - n} x^{\mu} 2^{2n-\mu}.$$

Setting $s=2$ in the Lemma and equating coefficients of x the desired result follows.

Solved also by T. W. Anderson, Jr., Shepard Bartnoff, J. R. Britton, Howard Eves, A. M. Glicksman, G. Grossman, J. F. Hofmann, M. Home,

J. B. Kelly, M. S. Macphail, C. D. Olds, G. W. Petrie, E. D. Schell, N. C. Scholomiti, E. R. Smith, G. Szegö, and the proposer.

Editorial Note. Macphail and Scholomiti's solutions are essentially as follows: Set $(x+1)^{(-j)} = 1/(x+1)(x+2)\cdots(x+j)$, then $\Delta^m(x+1)^{(-1)} = (-1)^m m!/(x+1)^{(-m-1)}$; and for $x=0$, we have $\Delta^m(0+1)^{(-1)} = (-1)^m m!/(m+1)! = (-1)^m/(m+1)$. Hence

$$\sum_{x=0}^{n-1} \frac{1}{x+1} = \left(\sum_{x=0}^{n-1} U^x\right)\frac{1}{0+1} = \frac{U^n-1}{U-1}(0+1)^{(-1)} = \frac{(\Delta+1)^n-1}{\Delta}(0+1)^{(-1)}$$

$$= \sum_{j=1}^{n} {}_nC_j\Delta^{j-1}(0+1)^{(-1)} = \sum_{j=1}^{n} {}_nC_j(-1)^{j-1}/j.$$

This symbolic reasoning may be reversed.

Schell stated that this problem appears in Chrystal's Algebra, vol. 2, p. 9, Ex. 18; and Olds stated that a solution was given by Greenstreet, *Mathesis*, ser. 2, vol. 1, 1891, p. 104 using a generating function; also in Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. 1, p. 6, prob. 38. About half of the remaining solvers used a generating function and the rest used difference methods in various forms.

This problem is related to the probability average problem 3652 [1933, 610], proposed by A. F. Stevenson, University of Toronto, and solved 1935, pp. 118–123, which is as follows:

The practice of certain cigarette manufacturers of supplying playing cards, poker hands, *etc.*, with their cigarette packets, and of offering various articles in exchange for complete set of these cards, suggests the following problem:

Assuming that each packet of cigarettes contains one of a set of 52 cards, and that these cards are distributed among the packets at random (the number of packets available being infinite), what is the average minimum number of packets that must be purchased in order to obtain a complete set of cards?

Three solutions were printed with the result $52\sum_{n=1}^{52} 1/n$ as the required average, that is 236 purchases. It is interesting to compare the methods used. In the solution of 4108 [1945, 282] Olds cited a related probability problem.

Factorial Binomial Theorem

4120 [1944, 289]. *Proposed by F. J. Duarte, Caracas, Venezuela*

Given k and h as positive integers, show that

$$\sum_{n=0}^{h-1}(-1)^n \frac{(h-1)(h-2)\cdots(h-n)}{n!}\,\frac{k(k+1)\cdots(k+h-n-1)}{(h-n)!} = \binom{k}{h},$$

where for $n=0$ the summand is $k(k+1)\cdots(k+h-1)/h!$.

Solution by G. T. Williams, Student, Harvard University. This is essentially the known factorial binominal theorem

$$(a+b)_{(k)} = \sum_{n=0}^{k} \binom{k}{n} a_{(n)} b_{(k-n)},$$

where $a_{(n)} = a(a-1)\cdots(a-n+1) = (-1)^n(n-a-1)_{(n)}$. Writing $\sum$ for the sum in the left-hand member of the problem, we have

$$k!\sum = \sum_{n=0}^{k-1} (-1)^n \binom{k}{n} (k-1)_{(n)}(-1)^{k-n}(-k)_{(k-n)}$$
$$= (-1)^k(k-k-1)_{(k)} = k_{(k)}.$$

This gives the desired result.

Solved also by H. S. M. Coxeter, J. B. Kelly, M. S. Knebelman, and Norman Miller.

Editorial Note. The factorial binomial theorem is easily proved. Let b have any fixed value and let a, r denote positive integers $a \geqq r$. Then we have

$$(a+b)^{(r)} = U^a b^{(r)} = (\Delta+1)^a b^{(r)} = \sum_{j=0}^{r} {}_aC_j \Delta^j b^{(r)},$$

where $b^{(r)} = b(b-1)\cdots(b-r+1)$, $b^{(0)} = 1$, and $\Delta^j b^{(r)} = r^{(j)} b^{(r-j)}$, the latter being zero if $j > r$. Hence

$$(a+b)^{(r)} = \sum_{j=0}^{r} \frac{a^{(j)}}{j!} r^{(j)} b^{(r-j)} = \sum_{j=0}^{r} {}_rC_j a^{(j)} b^{(r-j)}.$$

Consider now the two polynomials of degree r in a in the first and third expressions above, where a denotes any variable. They are equal for all integral values of $a \geqq r$, and hence they are equal for any value of a.

Another method of proof follows which applies to the ordinary binomial theorem. The theorem is true for $r = 1$; assume it true for a chosen r. Multiply each member of the assumed equation by $a+b-r$ giving on the left $(a+b)^{(r+1)}$, whereas on the right we set $a+b-r = (a-j)+(b-r+j)$ giving the addition of two sums

$$(a+b)^{(r+1)} = \sum_{j=0}^{r} {}_rC_j a^{(j+1)} b^{(r-j)} + \sum_{j=0}^{r} {}_rC_j a^{(j)} b^{(r-j+1)}.$$

Using ${}_rC_{j-1} + {}_rC_j = {}_{r+1}C_j$ we have

$$(a+b)^{(r+1)} = \sum_{j=0}^{r+1} {}_{r+1}C_j a^{(j)} b^{(r+1-j)},$$

and this completes the induction proof.

The problem identity may now be established without using the binomial theorem but by applying the two methods above. For the first method let h denote a positive integer, m a chosen non-negative integer, and k a number of any kind. Then we have

$$(m-k)^{(h)} = U^m(0-k)^{(h)} = (\Delta+1)^m(0-k)^{(h)} = \sum_{j=0}^{m} {}_mC_j\Delta^j(0-k)^{(h)}$$

$$= \sum_{j=0}^{m} {}_mC_j h^{(j)}(-k)^{(h-j)} = \sum_{j=0}^{m} {}_hC_j m^{(j)}(-k)^{(h-j)}.$$

In the final equality above set $m=h-1$ and we get

$$\sum_{j=0}^{h-1} {}_hC_j(h-1)^{(j)}(-k)^{(h-j)} = (h-1-k)^{(h)},$$

which easily reduces to the desired result.

The solution by Miller uses the induction proof as given above. Kelly denoted the given sum by $u_{k,h}$ and derived the difference equation $u_{k+1,h}-u_{k,h}=u_{k,h-1}$ and boundary conditions; these suffice to prove the desired result. The remaining solvers equated the coefficients of x^h in

$$(1+x)^{h-1}(1+x)^{-k} = \sum_{n=0}^{h-1}\binom{h-1}{n}x^n \sum_{j=0}^{\infty}\binom{-k}{j}x^j.$$

A second binomial formula is obtained by similar methods

$$(a+b)^{(-r)} = \sum_{j=0}^{a}\binom{-r}{j}a^{(j)}b^{(-r-j)}$$

where a, r are non-negative integers and b is any kind of number excluding integral values in the interval $-(r+a-1)\leqq b\leqq 0$, $b^{(-r)}=1/b(b+1)\cdots(b+r-1)$, $b^{(-0)}=1$. From this follows

$$(a+c)^{(a)} = \sum_{j=0}^{a}\binom{-r}{j}a^{(j)}(a+c+r)^{(a-j)}.$$

This last result follows also at once from the first binomial formula replacing in the left-hand member $a+c$ by $-r+(a+c+r)$, and simple alterations on the right.

Simple Continued Fractions

4123 [1944, 290]. *Proposed by V. Thébault, Tennie, Sarthe, France*

For what values of m is the product $(2m+1)(10m+1)$ a square of an integer? *Application.* In what systems of numbers is a number of the form *aabb* the square of a number of two digits *bb*?

Solution by M. F. Smiley, Annapolis, Md. If $(2m+1)(10m+1)$ is a square, then each of the two factors is a square, since they are odd and relatively prime. We set $x^2=2m+1$, $y^2=10m+1$ to obtain $5x^2-y^2=4$. The transformation (of determinant unity)

$$x = t+2u, \qquad y = 2t+5u,$$

yields the equation $t^2-5u^2=4$ (see Dickson's *Introduction to the Theory of Numbers*, 1929, p. 114). Note that $x\equiv y\equiv t\equiv u$ (2) so that we must delete from the sets (t_k, u_k) given by

$$(1) \qquad (t_k + u_k\sqrt{5})/2 = \left(\frac{3+\sqrt{5}}{2}\right)^k, \qquad k = \pm 1, \pm 2, \cdots,$$

those in which $t\equiv 0$ (2). To obtain an explicit formula for the deleted solutions, write $u=2U$ to obtain $t^2-20U^2=4$, whose general solution is given by

$$(t_j + U_j\sqrt{5})/2 = (9 + 2\sqrt{5})^j, \qquad j = \pm 1, \pm 2, \cdots$$

Since $(9+4\sqrt{5})=[(3+\sqrt{5})/2]^5$, we find that if $k\not\equiv 0$ (3) in (1), then $t\not\equiv 0$ (2) and

$$m = 2t_ku_k + (9u_k^2 + 3)/2, \qquad k = \pm 1, \pm 2, \cdots$$

gives the solution of our problem. For $t_1=3, u_1=1$, we get $m=12$, $(2m+1)(10m+1) = 5^2 11^2$; for $t_2=7$, $u_2=3$, $m=84$, the product is $13^2 29^2$; *etc.*

Application (sic). Let $q>1$ be the radix. Then $a, b<q$ and

$$aq^3 + aq^2 + bq + b = b^2(q+1)^2$$

yields

$$aq^2 + b = b^2q + b^2.$$

Setting $b^2=mq+n$ $(m, n<q)$ gives

$$aq^2 + b = mq^2 + (m+n)q + n.$$

Hence $n=b$, $m+n=q$, $a=q+1-b$, and $b^2=(q-b)q+b$. Setting $t=5b-2$, $u=2q-b$ in this last equation gives $t^2-5u^2=4$ again. The requirement that b and q be integers is equivalent to $t\equiv -2(5)$ as is easily seen. The least positive solution $(t, u)=(3, 1)$ satisfies this requirement but not $q>1$. It is quite simple to check that if in (1) we have $t_k\equiv -2(5)$, then $t_{k+1}\equiv 2(5)$; while if in (1) we have $t_k\equiv 2(5)$, then $t_{k+1}\equiv -2(5)$. Hence the formula

$$(t_k + u_k\sqrt{5})/2 = [(3+\sqrt{5})/2]^k, \qquad k = 3, 5, 7, \cdots$$

yields all values of (t, u) and thus (q, b) which satisfy our problem. The first four solutions are

t	18	123	843	5778
u	8	55	377	2584
b	4	25	169	1156
q	6	40	273	1870
a	3	16	105	715.

Solved also by Murray Barbour, H. N. Carleton, J. B. Kelly, E. P. Starke, and the proposer.

Editorial Note. The proposer and Barbour derived also non-integral values $m=3/2, 1155/2, \cdots$. To obtain in turn the solutions of $y^2-5x^2=-4$, Starke used the relations

$$y' = |9y \pm 20x|, \qquad x' = 9x \pm 4y,$$

where $(x, y)=(1, 1)$ gives $(x', y')=(5, 11), (13, 29)$; and referred to 4047 [1944, 103]. Barbour referred to Dickson's *History of the Theory of Numbers*, vol. 2, chaps. XII, XIII, in particular to the remarks on the work of S. Realis, p. 407. The solutions of $5t^2-4=\square$ are found to be $t=1, 2, 5, 13, 34, 89, \cdots$ which is the Fibonacci series $S_n=3S_{n-1}-S_{n-2}$. The required base r is the product of successive pairs of the additive series 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, $\cdots$ $r=1\cdot 1, 2\cdot 3, 5\cdot 8, 13\cdot 21, \cdots$. Kelly also referred to the Fibonacci series.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending new items to B. W. Jones, White Hall, Cornell University, Ithaca, N. Y.

The National Research Council announces predoctoral fellowships in the mathematical, physical and biological sciences. These fellowships are intended to assist young men and women, whose graduate study has been prevented or interrupted by the war, to complete their work for the doctorate. The annual stipend will be $1,200 for single persons and $1,800 for married men plus an allowance for tuition fees. Those interested should write to the Secretary, Committee on Predoctoral Fellowships, National Research Council, 2101 Constitution Avenue N.W., Washington 25, D. C.

Professor H. M. Gehman, branch head of the Shrivenham American University in England, announces the following civilian staff: V. W. Adkisson, J. P. Ballantine, T. C. Benton, E. T. Browne, F. J. H. Burkett, D. R. Davis, W. M. Davis, P. D. Edwards, C. J. Latimer, L. L. Lowenstein, J. H. Neelley, P. R. Rider, N. E. Rutt, C. G. Stipe, J. I. Tracey. It is hoped that in a later number announcement can be made of the staff in each of the other European army university centers.

G. E. Stechert and Company have received a cable from Professor H. Steinhaus of Krakow informing them that he is well and that his address is Lidia Kott, Krakow Lea 93, Poland.

At the University of California the following promotions are announced: Assistant Professor A. L. Foster to an associate professorship; Dr. L. H. Swinford to an assistant professorship; Dr. Alfred Tarski to an associate professorship.

Associate Professor C. K. Alexander of Occidental College has been promoted to a professorship.

Dr. Florence E. Allen of the University of Wisconsin has been promoted to an assistant professorship.

Dr. K. J. Arnold has been appointed to an assistant professorship at the University of Wisconsin.

Assistant Professor R. A. Beaver of the New York State College for Teachers, Albany, has been promoted to a professorship.

Associate Professor H. F. Bohenblust of Princeton University has been appointed to a professorship at Indiana University.

Assistant Professor E. C. Brown of Worcester Polytechnic Institute has been promoted to a professorship.

Associate Professor O. E. Brown of Case School of Applied Science has been promoted to a professorship.

Associate Professor R. E. Brown of Rhode Island State College has been promoted to a professorship.

Eleanor Calkins of the College of William and Mary has been promoted to an assistant professorship.

Associate Professor Teresa Cohen of Pennsylvania State College has been promoted to a professorship.

Dr. W. J. Combellack of Northeastern University has been promoted to an assistant professorship.

Associate Professor A. T. Craig of the University of Iowa has been promoted to a professorship.

Dr. A. W. Davis of Iowa State College has been promoted to an assistant professorship.

James Edgar Davis has been appointed to an assistant professorship at the University of Illinois.

Assistant Professor F. L. Dennis of Ursinus College has been promoted to an associate professorship.

Professor C. E. Dimick of the U. S. Coast Guard Academy has retired.

N. E. Dodson of Newberry College, South Carolina, has been promoted to an associate professorship.

Assistant Professor W. C. Doyle of Rockhurst College, Kansas City, Missouri, has been promoted to an associate professorship.

Assistant Professor L. T. Dunlap of Pennsylvania State College has been promoted to an associate professorship.

Assistant Professor R. H. Fox of Syracuse University has been appointed to an assistant professorship at Princeton University.

Assistant Professor L. M. Garrison of Louisiana Polytechnic Institute, Ruston, Louisiana, has been promoted to an associate professorship.

Dr. G. D. Gore has been appointed to a professorship and the chairmanship of the department of mathematics and engineering science at Roosevelt College, Chicago, Illinois.

Dr. J. W. Green has been appointed to an assistant professorship at the University of California at Los Angeles.

Professor H. H. Hartzler of Goshen College has been appointed to a professorship at Bluffton College, Bluffton, Ohio.

Lieutenant C. C. Hurd of the U. S. Coast Guard Academy has been appointed dean of Allegheny College.

Assistant Professor J. L. Kelley of the University of Notre Dame has been appointed to an assistant professorship at the University of Chicago.

Assistant Professor J. C. C. McKinsey of Montana State College has been appointed to an assistant professorship at the University of Nevada.

Dr. W. A. Mersman of the College of Agriculture of the University of California has been promoted to an assistant professorship.

Dr. R. v. Mises of Harvard University has been appointed Gordon McKay professor of aerodynamics and applied mathematics.

Dorothy J. Morrow of Bryn Mawr College has been appointed to an assistant professorship at George Washington University.

Professor L. W. Stark of William Jewell College has been appointed acting professor of mathematics at Atlantic Christian College, Wilson, North Carolina.

Assistant Professor Alexander Weinstein of the University of Toronto has been appointed to an associate professorship.

Dr. Max Wyman of the University of Alberta has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Amherst College: Daniel Finkel, F. G. Graff, R. L. Wine
University of Arizona: Dr. H. S. Kieval
University of Illinois: Dr. J. S. Stubbe

GENERAL INFORMATION

EDITED BY C. V. NEWSOM

Send information of especial interest to mathematicians, exclusive of personal items, to C. V. Newsom, Oberlin College, Oberlin, Ohio.

THE COOPERATIVE COMMITTEE ON SCIENCE TEACHING

The Cooperative Committee on Science Teaching is a committee of the American Association for the Advancement of Science. Each scientific organization interested in the project nominates one or more representatives to serve upon the Committee; final appointment, then, is made by the Executive Committee of the A.A.A.S. At the present time, Dr. R. J. Havighurst, division of human relations, University of Chicago, is chairman of the Committee, and mathematicians are represented by Professor R. W. Schorling, University of Michigan (M.A.A.), and by Professor E. H. C. Hildebrandt, Northwestern University (National Council).

A preliminary report upon the work of the Cooperative Committee was published in *School Science and Mathematics*, October, 1942. The tentative proposals and recommendations of that article are still under discussion. Mathematicians concerned with the problem of secondary teaching will follow the work of the Committee with interest.

The problem with which the Committee has been especially concerned is the preparation of teachers for the small high school. The Committee has found that half of the high schools in the country have five teachers or less. Three-fourths of them have ten teachers or less. Yet the small school must offer courses in at least twelve subjects, and many of them offer fifteen to twenty subjects. Moreover, most new teachers commence their work in small high schools because of the fact that the larger schools require teaching experience of candidates for positions. The beginning teacher in a small school nearly always must teach at least three different subjects, and often four or five.

Consequently, the Cooperative Committee believes that the science and mathematics curriculum in the small high school, as well as the preparation of teachers properly trained to handle such a program, represents an urgent problem for investigation. It is planned in the near future to make some specific recommendations to colleges and universities training prospective secondary teachers.

A COURSE FOR PROSPECTIVE TEACHERS IN OHIO

A committee of the Ohio Section of the M.A.A., under the chairmanship of Professor I. A. Barnett of the University of Cincinnati, has completed a tentative course of study in mathematics for prospective teachers in the elementary schools of Ohio. The general objectives of the course of study have been described in the following terms: "To present to teachers the historical and logical

background requisite to the proper and interrelated understanding of the appropriate mathematical facts and processes; to feature the human significance of this material and its usefulness in concrete applications; and to integrate the insight and perspective thus gained with the teaching process."

Five special objectives are listed as follows:

1. *Symbolism.* To emphasize the basic importance of mathematical symbolism; symbols as a powerful shorthand of thought—their role in grasping and solving difficult problems.

2. *Vocabulary.* There is no more important goal in teaching mathematics than in training the students to talk accurately in the mathematical classroom. This attainment will facilitate immeasurably the teaching and understanding of mathematics. The lack of practice on the part of the student in talking mathematics accurately means necessarily that much of what is going on in the classroom has little meaning to the student.

3. *Justification of the processes of arithmetic.* Review of arithmetic with emphasis on the reasons for the underlying processes as well as on the attainment of mechanical proficiency.

4. *Arithmetic-based algebra.* To show that the consciousness of algebraic thinking naturally dawns and develops in arithmetic.

5. *Geometry.* The intuitive is to go hand in hand wherever possible with the logical. The training of the logical faculty is not dependent on the quantity of theorems. It must be assured through independent thinking, and, to attain this important end, quantity should be drastically sacrificed if necessary.

The instructor of such a training course has been given three suggestions; namely,

1. The presentation should be informal and favor methods which teachers could later use in their own classrooms. The student teachers should have a major part in the development of the course.
2. It is very desirable that the definitions should be related to concrete experience rather than merely quoted from the books. The process of generalization, one of the most important concepts in the whole domain of mathematics, is to be given prominence through precept and exercise.
3. The teacher's interest is to be stimulated by stressing the part played by the mathematics of the past in the development of civilization. Mathematics is to be viewed as a living and growing force in our daily lives and not as a dead and closed chapter of the past.

Anyone interested in the actual content of the proposed course should write to Professor Rufus Crane, Ohio Wesleyan University, Delaware, Ohio, to obtain a mimeographed copy of the tentative outline.

ACADEMIC INTERESTS OF DISCHARGED VETERANS

All surveys which have been made of the academic interests of discharged veterans who are anticipating a return to college indicate a pronounced trend toward the technical fields. Moreover, it is becoming increasingly evident that

most of the men are thinking only of those branches of engineering and technical science most obviously related to the conduct of the war. In fact, Mr. B. A. Thresher, Director of Admissions at the Massachusetts Institute of Technology, has warned of a possible dislocation of the nation's educational programs and technical needs if present trends are ignored.

Mr. Thresher has analyzed 1,102 applications for admission to M.I.T., which he has recently received from men in service. The survey indicates that the majority of prospective students interested in engineering are thinking only of electrical, mechanical, and aeronautical engineering. Moreover, there appears to be very little interest generally in such fields as biology, geology, mathematics, and architecture. Even such fields as chemistry and metallurgy are being neglected by the returning veteran.

Mr. Thresher sampled the applications to obtain some specific data upon the interest in mathematics. A prewar sample showed that 1.2% of the enrollment in the institution was in mathematics. This included both undergraduate and graduate work. In the sample of applications for postwar admission, 0.67% showed an interest in mathematics. The numbers involved in the sample were perhaps too small to permit any broad conclusions. The figures on mathematics, however, may be contrasted with a comparable sample in the case of electrical engineering which shows an increase in postwar registrations of over 150%; aeronautical engineering shows almost as great an increase. In summary, Mr. Thresher writes, "It appears that mathematics is one of several fields which are, relatively speaking, being neglected by men in the service because they have been in close contact with the particular branches of electronics and aeronautics."

Applicants generally, according to Mr. Thresher, seem to be interested in "narrow and intensive training." Moreover, they fail to realize the importance of basic subjects.

TEACHERS FOR THE VETERAN REHABILITATION PROGRAM

The Veteran Rehabilitation Program is offering opportunities for recreational workers, physical directors, and teachers of both academic and commercial subjects to aid in the adjustment of disabled war veterans to civilian life. According to Announcement No. 362 (revised August 13, 1945), the Civil Service Commission is seeking qualified persons for these positions. No written test is required. The salaries are $2,190 and $2,433 a year, including overtime pay. Information may be obtained by writing directly to the Civil Service Commission, Washington 25, D. C.

To qualify for this work, applicants must have had at least one year of responsible experience in teaching mathematics or other academic subjects. In addition, applicants must have successfully completed a full four-year course of study leading to a bachelor's degree in a college, university, or teacher-training institution of recognized standing, with a minimum of twelve semester hours in education, including at least one course in practice teaching, provided that an

additional year of the experience described above may be substituted for the required course in practice teaching.

A teacher selected for the Program will teach an academic subject or a combination of such subjects to patients in Veterans Administration hospitals upon recommendation of the attending physician. He also will plan, organize, and schedule classes for patients or arrange for individual instruction, and will make recommendations to the manager regarding the use of correspondence courses and make the necessary arrangements for such courses. He will be expected to coordinate both class and individual instruction with the vocational rehabilitation service.

IMPORTANT LEGISLATION BEFORE CONGRESS

In view of the publicity given to the report, scientists generally are familiar with the recommendations of Dr. Vannevar Bush, Director of the Office of Scientific Research and Development, in regard to the role of the federal government in supporting and encouraging scientific research. The complete report is entitled *Science, The Endless Frontier*, and it may be obtained from the Superintendent of Documents, Government Printing Office, Washington 25, D. C. The price is thirty cents per copy.

As soon as the Bush report was released, three identical bills were introduced into Congress following its recommendations. They were (S. 1285) by Senator Warren G. Magnuson, Washington, (H.R. 3582) by Representative Wilbur D. Mills, Arkansas, and (H.R. 3860) by Representative Jennings Randolph, West Virginia. The Senate bill was referred to the Committee on Commerce and the House bills to the Committee on Interstate and Foreign Commerce.

In addition to the recommendations of Dr. Bush, another report bearing upon the same general question was made on July 23 by the Subcommittee on War Mobilization of the Senate Military Affairs Committee under the chairmanship of Senator Harley M. Kilgore of West Virginia.

The Kilgore report states that "We have depended in the past very heavily on the basic research done in Germany with the support of the German government. Through the cartel system we have also presumably relied on Germany for a great deal of our applied research. It is quite clear now that we can no longer rely upon Germany for basic or applied research. It will be necessary to reduce the German leadership in science and technology which time after time has led to wars of aggression. Hereafter we must rely on ourselves for basic research. This is a field of public responsibility, because basic research is seldom immediately profitable, although the application of such research may be of enormous value. The Nation's universities, both before and during the war, have distinguished themselves for their work in basic research. Congress should provide for a broad and representative board to advise on the use of the university laboratories and of some government facilities in a well-planned program of basic research. Funds must be provided to carry on the work upon which our future progress in applied science and technology depends."

To implement the provisions of its report, the Kilgore Committee recommends the establishment of a National Science Foundation. It would be headed by a director appointed by the President and a National Science Board under the chairmanship of the Director. The board would be made up of the Secretaries of War, Navy, Interior, Agriculture, Commerce, and Labor, the Attorney General, the Federal Security Administrator, and eight members appointed by the President to represent the public.

The Foundation itself would not be a research agency, but it would have the function of stimulating scientific research at all levels and would encourage the development and training of scientific personnel. It is further recommended that 20 per cent of the funds appropriated shall be used for national defense research, 20 per cent for medical research, and the remaining 60 per cent for basic science and other research.

Senators Kilgore, Edwin C. Johnson of Colorado, and Claude Pepper of Florida are the sponsors of a bill (S. 1297), to establish a National Science Foundation as recommended by the Committee. The bill has been referred to the Committee on Military Affairs.

As a further consequence of the Bush and Kilgore reports, Senator Thomas of Utah introduced S. 1316 to give federal aid to the states for science education. The bill asserts that there is a "widespread lack of opportunity for youth enrolled in public secondary schools to receive effective instruction in natural science subjects." So the bill proposes to appropriate $4,000,000 in 1945, increasing to $20,000,000 by 1951, to help the states pay salaries of supervisors and teachers of natural sciences and to help defray the cost of supplies and equipment. This money would be allotted to the states on the basis of the secondary-school population, and all funds would be expended "through public agencies and under public control as determined by the legislatures of the respective states." The Thomas bill has been referred to the Committee on Education and Labor.

An Interesting Theorem. *All real numbers are uninteresting.* For the integer one is evidently uninteresting. (It has only trivial representations a sum of Mth powers of integers, and trivially divides every other number.) Now assume that there are interesting integers, and let N be the least such. Then N is greater than one. Hence $N-1$ is a *very* interesting integer; for it is the first uninteresting integer whose immediate successor is not interesting. This contradiction shows that no positive integer is interesting. But every positive rational is trivially expressible in an infinite number of ways as an ordered couple of uninteresting numbers and hence is uninteresting. Since every negative rational may be obtained from an uninteresting number by a mere change of sign, no negative rational is interesting. But zero and each of the remaining real numbers is representable as the least upper bound of a set of uninteresting numbers . . . . We conclude that no real numbers are interesting. The extension to complex numbers and Hilbert space is left as an exercise to the reader.—*Morgan Ward.*

THE MATHEMATICAL ASSOCIATION OF AMERICA

NEW MEMBERS

The following fifty persons have been elected to membership on applications duly certified:

REV. J. H. ANTONY, A.B. (Univ. of Dayton) Head of Dept., Purcell High School, Cincinnati, Ohio

J. G. BACHMANN, B.S. (New York Univ.) President, Bachmann and Co., Inc., 8511 Sunset Blvd., Hollywood 46, Calif.

T. H. BEDWELL, A.B. (South Dakota) Instr., Physics, Univ. of South Dakota, Vermillion, S. D.

K. H. BRACEWELL, Ph.D. (Indiana) Prof., Math. and Physics, Hamline Univ., St. Paul, Minn.

T. S. BRODERICK, M.A., F.T.C.D. (Dublin) Prof., Trinity Coll., Dublin, Ireland

R. H. BRUCK, Ph.D. (Toronto) Asst. Prof., Univ. of Wisconsin, Madison 6, Wis.

L. H. CHACALOS. Senior in High School, Orange, N. J.

P. L. CHESSIN. Student, Univ. of Wisconsin, Madison, Wis.

M. P. DE REGT, B.S. (Webb Inst.) Supervisor, Hull Tech. Section, Todd-Pacific Shipyards, Tacoma, Wash.

J. A. H. DUFFIE, B.S. (Univ. of New Brunswick) Grad. student, Chemistry-Physics, Catholic Univ. of America, Washington 17, D. C.

REV. L. E. ERNSDORFF, M.S. (Univ. of Notre Dame) Instr., Loras Coll., Dubuque, Iowa

M. P. GAFFNEY, B.S. (Harvard) C Sp (x), U.S.N.R. Naval Research Lab., Washington, D. C.

H. W. GODDERZ, B.S. (Northern St.T.C.) Coll. of St. Thomas, St. Paul 1, Minn.

ARTHUR GRAD, A.M. (Columbia) Mathematician, U. S. Coast and Geodetic Survey, Washington, D. C.

BROTHER JOSEPHUS GREGORY, A.M. (Loyola Univ.) St. Mary's Coll., Winona, Minn.

MARGARET M. HANSMAN, Ph.D. (Illinois) Asst. Prof., Colorado Coll., Colorado Springs, Colo.

R. G. HELSEL, Ph.D. (Ohio State) Instr., Ohio State Univ., Columbus, Ohio

F. W. HERLIHY, B.S. (Chicago) Vice-Pres., Herlihy Mid-Continent Co., Comstock, Mich.

D. L. HOLL, Ph.D. (Chicago) Prof., Iowa State Coll., Ames, Iowa

REV. J. A. HRATZ, B.S. (Iowa) Acting Head of Dept., St. Ambrose Coll., Davenport, Iowa

MEYER KARLIN, Ph.D. (Columbia) Instr., Brooklyn Coll. Evening Session, Brooklyn, N. Y.

NORBERT KAUFMAN. S 2/c, U. S. Navy

WINFIELD KECK, A.M. (Pennsylvania) Instr., Math. and Physics, Muhlenberg Coll., Allentown, Pa.

NATHAN KEYFITZ, B.S. (McGill Univ.) Statistician, Dominion Bureau of Statistics, Ottawa, Ont., Can.

R. J. KOCH. RT 3/c, U. S. Navy

WALTER LEIGHTON, Ph.D. (Harvard) Lecturer, Rice Inst., Houston, Tex. On leave. Dir., Applied Math. Group, Northwestern Univ.

ANNE L. LEWIS, Ph.D. (Chicago) Asst. Prof., Woman's Coll., Univ. of North Carolina, Greensboro, N. C.

REV. C. J. LEWIS, M.S. (Georgetown Univ.) Instr., St. Peter's Coll., 144 Grand St., Jersey City 2, N. J.

SISTER M. MECHTILDIS MALIN, M.S. (Marquette) Instr., Viterbo Coll., LaCrosse, Wis.

SISTER MARY MICHAEL MALONEY, A.M. (Catholic Univ.) Head of Dept., D'Youville Coll., 320 Porter Ave., Buffalo 1, N. Y.

C. R. MUMMERY, B.S. (Toronto) Head, Mech. and Cleaner Research Sec., The Hoover Co., North Canton, Ohio

SISTER NOEL MARIE, A.M. (Catholic Univ.) Head of Dept., Coll. of Saint Rose, Albany 3, N. Y.

P. L. NORIEGA. Student, Geneva Coll., Beaver Falls, Pa.

F. F. OTIS, B.Ed. (St. T.C., River Falls, Wis.) Instr., Engg., Marquette Univ., Milwaukee, Wis.

ELLI OTTESON, A.M. (Wisconsin) Chm. of Dept., High School, Eau Claire, Wis.

S. T. PARKER, M.A. (Univ. of British Columbia) Asst. Prof., Speed Scientific School, Univ. of Louisville, Louisville 8, Ky.

MURRAY PESHKIN. T/4, U. S. Army

A. P. RHODES, A.M. (Stanford) Asso. Civil Engr., U. S. Navy, Dept. of Public Works, San Diego, Calif.

ETHEL ROORDA, M.S. (Iowa) Instr., Buena Vista Coll., Storm Lake, Iowa

EDWARD ROSENTHALL, Ph.D. (Calif. Inst. of Tech.) Lecturer, McGill Univ., Montreal, P. Q., Can.

HARRY SCHOR, M.S. in Educ. (C.C.N.Y.) Chm. of Dept., Manhattan High School of Aviation Trades, New York, N. Y.

C. F. SECADA, B.S. (Peruvian Naval Acad.) Grad. student, Massachusetts Inst. of Tech., Cambridge 39, Mass. Lt. Comdr., Peruvian Navy

AARON SHAPIRO, A.M. (C.C.N.Y.) Teacher, Midwood High School, Brooklyn, N. Y.

L. R. SHOBE, M.S. (Kansas State Coll.) Tech. Instr., General Motors Inst., Flint 2, Mich.

ETHEL SUTHERLAND, Ph.D. (Columbia) Instr., East Carolina Teachers Coll., Greenville, N.C.

ANNITA TULLER, Ph.D. (Bryn Mawr) Instr., Hunter Coll., New York, N. Y.

R. Z. VAUSE, JR., B.S. (South Carolina) Radar Technician, U. S. Navy

R. D. WAGNER, Ph.M. (Wisconsin) Instr., Univ. of Wisconsin, Madison, Wis.

ETTA A. WAITE, A.M. (Columbia) Teacher, Girls Commercial High School, Brooklyn, N. Y.

H. E. WOODWARD, A.M. (Texas Tech.) Instr., Texas Tech. Coll., Lubbock, Tex.

W. B. CARVER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Twenty-ninth Annual Meeting, Chicago, Ill., November 24–25, 1945.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA, Berkeley, January 26, 1946
OHIO, April 4, 1946
OKLAHOMA
PHILADELPHIA, Philadelphia, December 1, 1945
ROCKY MOUNTAIN
SOUTHEASTERN
SOUTHERN CALIFORNIA, Pasadena, March 9, 1946
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE
WISCONSIN, Milwaukee, May, 1946

W. L. Hart's TRIGONOMETRIES

Plane and Spherical Trigonometry with Applications

A recently published, efficient and concise presentation with a strong focus on numerical applications and on practical problems. The trigonometry of the acute angle and logarithms are developed before the general angle. *222 p., $2.00. With Tables, 349 p., $2.25. Tables separately, 127 p., $1.35.*

Plane Trigonometry with Applications

177 p., $1.75. With Tables, 304 p., $2.00

Trigonometry, Plane and Spherical

Full recognition is given to both the numerical and analytical aspects, with greater stress on the theoretical phases than is found in Hart's *Plane and Spherical Trigonometry with Applications.* The acute angle is presented before the general angle. *225 p., $2.00. With Tables, 368 p., $2.25. Tables separately, 127 p., $1.35.*

Plane Trigonometry

207 p., $1.75. With Tables, 334 p., $2.00

D. C. HEATH AND COMPANY

BOOK NEWS

Raymond W. Brink's

ALGEBRA: COLLEGE COURSE

Supplies the materials for a complete and rich course in college algebra for students who are not in need of a review of high-school higher algebra. 8vo, 329 pp. $2.15.

COLLEGE ALGEBRA

Presents all the material in ALGEBRA: COLLEGE COURSE with the addition of a systematic review of high-school higher algebra. 8vo, 445 pp. $2.40.

INTERMEDIATE ALGEBRA

Presents by itself the systematic review of high-school higher algebra included in COLLEGE ALGEBRA. 8vo, 268 pp. $1.50.

D. APPLETON-CENTURY COMPANY
35 West 32nd Street New York 1, New York

New Texts from McGraw-Hill

INTERMEDIATE ALGEBRA

By R. ORIN CORNETT, Harvard University, on leave from Oklahoma Baptist University. 317 pages, $2.00

An entirely new approach. The text is based on the idea that mastery of detail and routine operations should be preceded by an understanding of purpose, significance, and relation to the whole. A sharp distinction is made between the algebraic method and the algebraic operations which serve as tools used in applying the method.

COLLEGE ALGEBRA. *New second edition*

By PAUL K. REES, Southwestern Louisiana Institute, and FRED W. SPARKS, Texas Technological College. 385 pages, $2.50

A revision of this well known textbook, offering a logically arranged, stimulating treatment of the topics usually covered in the freshman course in College Algebra. The new edition stresses review material, and contains a wealth of completely new problems.

ANALYTIC GEOMETRY

By Ross R. MIDDLEMISS, Washington University. 306 pages, $2.75

Designed especially to meet the needs of scientific and technical students. Conic sections are treated somewhat more briefly than usual. There is a chapter on the polynomial curves and more than the usual amount of space on the exponential, logarithmic, trigonometric, and inverse trigonometric functions, and their graphs.

INDUSTRIAL ALGEBRA AND TRIGONOMETRY. With Geometrical Applications

By JOHN H. WOLFE, formerly of Ford Apprentice Schools, and WILLIAM F. MUELLER and SEIBERT D. MULLIKIN, Ford Apprentice Schools. 275 pages, $2.20

Presents a practical and more extended application of trigonometric principles than has hitherto been available, with emphasis on the problems occurring in the mechanical and electrical engineering departments of industry. The text includes linear, quadratic, and higher degree equations, Horner's method and compound angles, etc.

THE DEVELOPMENT OF MATHEMATICS. *New second edition*

By E. T. BELL, California Institute of Technology. 618 pages, $5.00

In this revision of a well known and widely used book, the author tells with freshness and vigor the absorbing story of the part played by mathematics in the evolution of civilization, from about 4000 B.C. to the present day. The second edition contains a wealth of new material, much of which is concerned with recent trends and developments in modern mathematics.

Send for copies on approval

McGRAW-HILL BOOK COMPANY, Inc.

330 West 42nd Street **New York 18, N.Y.**

COLLEGE TEXTS

GENERAL MATHEMATICS

By Currier, Watson & Frame. "The materials of the various subjects are well integrated. The explanations are concise and clear, and there are many worked examples. There is an abundance of problem material and there are many historical notes. . . . Abundant material for a year of mathematics."—*School Science and Mathematics.* 2nd Ed., $3.00.

COLLEGE ALGEBRA

By Paul R. Rider. "The book is carefully written and each chapter contains a liberal supply of graded problems. The chapter on theory of equations consists of rather more material than is to be found in many of the books of similar scope. Teachers interested in statistics will welcome the appearance of a chapter on finite differences."—*National Mathematics Magazine.* $2.00.

PLANE & SPHERICAL TRIGONOMETRY

By Paul R. Rider. "The book is written in flexible units so that it can be adapted to different types of courses. But more important than this is the clear and careful exposition, well selected illustrative examples, and excellent lists of exercises."—*American Mathematical Monthly.* With tables, $2.50. Without tables, $2.00.

DIFFERENTIAL & INTEGRAL CALCULUS

By Clyde E. Love. "One of the most widely used calculus texts."—*School Science and Mathematics.* "A very usable text. . . . The material is carefully selected and organized. The entire presentation keeps the student well in mind."—*Peabody Journal of Education.* 4th Ed., $3.25.

ANALYTIC GEOMETRY

By Clyde E. Love. "The third edition of this successful text retains the good features of the former editions. In addition the subject matter has been reorganized so as to treat the conic sections more effectively than in previous editions and at the same time require much less space. . . . The introduction to Solid Analytic Geometry is one of the best to be found in any elementary text. The exposition throughout is brief and to the point—a characteristic of all of Love's texts."—*School Science and Mathematics.* 3rd Ed., $2.75.

THE MACMILLAN COMPANY
60 FIFTH AVENUE NEW YORK 11, N. Y.

Mathematics In Human Affairs

by F. W. Kokomoor

☆ Arithmetic, algebra, geometry, trigonometry, analytic geometry, mathematics of finance, statistics, and calculus are covered in this interesting mathematics survey. Intended for freshmen, the book covers fundamental general mathematics in a manner that is interest-awakening and exciting, yet always accurate.

Illustrations from real life situations are used to introduce various problems. This tie-up impresses upon the student the daily applicability of mathematics. Historical connections and descriptive material are employed to make principles clear; a study guide with suggestions, questions, problems, and exercises aid permanent learning.

Technical mathematics is kept to a minimum. The text will afford a person with almost no previous preparation a well-rounded understanding of the subject.

754 pages College List $3.75

Principles of College Algebra

by T. Y. Thomas and M. S. Knebelman

☆ A basic one-semester course in college algebra, this text teaches quickly and thoroughly. It is also adaptable to a full-year course.

The subject is developed gradually and methodically from a few basic concepts through the principles and fundamentals of algebra. The emphasis is on precise definitions and logical development of their consequences. All historical and recreational material has been omitted to permit the student to progress more easily and rapidly.

380 pages College List $2.85

The Elements of Statistics

by Elmer B. Mode

☆ Intended for use in mathematics courses, this text covers clearly and simply the fundamentals of statistics. Numerous problems and illustrations are used, applying statistics to such fields as social sciences, business, sociology, and biology.

THE ELEMENTS OF STATISTICS requires no special training beyond high school mathematics. Many teacher aids are used, including line drawings, charts, graphs, tables, bibliographies, questions and answers, and footnotes. The arrangement is unusually flexible, and you will enjoy teaching from a text that allows so much latitude in supplementary material.

378 pages College List $3.50

Mathematics of Finance

by Simpson, Pirenian, & Crenshaw

☆ This text is designed primarily for a three-hour, one-year course for students of commerce and business administration. It is divided into two parts: the first provides a complete survey of the essentials of commercial algebra, with emphasis on the processes most useful in business; Part Two is an introduction to the mathematical theory of compound interest, annuities, life insurance, and statistics. (The latter part may be obtained separately.)

469 pages (with tables) College List $3.75
tables alone College List .50
Part Two, separate College List 2.50
answers, on teacher's request List .15

Send for approval copies

p-h **PRENTICE-HALL, INC.** 70 FIFTH AVENUE, NEW YORK 11

GEORGE BANTA PUBLISHING COMPANY, MENASHA, WISCONSIN, U.S.A.

UM

10

CHNOLOGY DEPT.

JAN 1 2 1946

THE AMERICAN MATHEMATICAL MONTHLY

DEVOTED TO THE INTERESTS OF
COLLEGIATE MATHEMATICS

VOLUME 52 NUMBER 10

PART I

CONTENTS

On the Nature of Mathematical Truth . . . . C. G. HEMPEL 543
Two Dimensions. . . . . J. L. COOLIDGE 557
An Elementary Constructive Proof of the Fundamental Theorem of Algebra . . . . . P. C. ROSENBLOOM 562
Discussions and Notes . . H. F. SANDHAM, WILLIAM FUNKENBUSCH 571
Recent Publications. 574
Problems and Solutions. 578
News and Notices 589
General Information 592
University Programs for Ex-Service Personnel in Canada . . . . . F. S. NOWLAN 592
The Availability of Meteorologists as Instructors 599
Mexican Mathematics. . . . . RUFUS OLDENBURGER 600
Marine Corps Criticizes the Teaching of Mathematics 601
The Mathematical Association of America 601
Calendar of Future Meetings 601
Index to Volume 52, 1945 602

DECEMBER 1945